MA22S3 Tutorial Sheet 3, outline solutions.[1,2]

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Questions

1. (4) Compute the Fourier transform of $f(t) = e^{-a|t|}$ where a is a positive constant.

   Solution: The idea here is to plug $f(t)$ directly in to the equation for $\tilde{f}(k)$ and then use $|t| = t$ for $t > 0$ and $|t| = -t$ for $t < 0$.

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{-ikt}\ f(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{-ikt}\ e^{-a|t|} = \frac{1}{2\pi}\left[\int_0^{\infty} dt\ e^{-ikt-at} + \int_{-\infty}^{0} dt\ e^{-ikt+at}\right] \\
&= \frac{1}{2\pi}\left[-\frac{e^{-t(a+ik)}}{a+ik}\Big|_0^{\infty} - \frac{e^{t(a-ik)}}{a-ik}\Big|_{-\infty}^{0}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \frac{1}{\pi}\frac{a}{a^2+k^2}.
\end{aligned}$$

   f can be represented as a Fourier integral

$$f(t) = \int_{-\infty}^{\infty} dk\ e^{ikt}\ \tilde{f}(k) = \frac{a}{\pi}\int_{-\infty}^{\infty} dk\ \frac{e^{ikt}}{a^2+k^2}.$$

2. (4) Express the following function as a Fourier integral:

$$f(x) = \begin{cases} \cos t & |t| < \frac{\pi}{2} \\ 0 & |t| > \frac{\pi}{2} \end{cases}$$

   One way to do the required integral is to split the cosine into exponentials.

   Solution: (a) Writing f as a Fourier integral $f(t) = \int_{-\infty}^{\infty} e^{ikt}\ \tilde{f}(k)\ dk$. We require the Fourier transform, the important thing is to note that the function is zero outside of $(-\pi/2, \pi/2)$ so the integral outside of that range doesn't contribute to the answer

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{-ikt}\ f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt\ e^{-ikt}\ \frac{e^{it}+e^{-it}}{2} \\
&= \frac{1}{4\pi}\left(\frac{e^{i(1-k)t}}{i(1-k)} + \frac{e^{i(-1-k)t}}{i(-1-k)}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-ik\pi/2}+ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2}-ie^{ik\pi/2}}{i(-1-k)}\right]
\end{aligned}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3

[2] Including material from Chris Ford, to whom many thanks.

$$= \frac{1}{4\pi}\ 2\cos\left(\frac{k\pi}{2}\right)\ \left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right)\ \frac{1}{1-k^2}.$$

Therefore

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\ \cos\left(\frac{k\pi}{2}\right)\ \frac{e^{ikt}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$.