MA22S3 Tutorial Sheet 1, outline solutions.[1]

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Questions

1. (2) Establish that

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \cos nt = 0 \tag{1}$$

for integers n and m.

Solution: This is an exercise in trigonometry and uses first the trignometric identity

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \cos nt = \frac{1}{2}\int_{-\pi}^{\pi} dt\ [\sin{(n-m)t} - \sin{(n+m)t})] = 0 \tag{2}$$

and then the fact that the integral of a trignometric function over its whole period is zero. This fact is used a lot; take sine for example, and let N be an integer. Consider

$$I = \int_{-\pi}^{\pi} \sin Nt dt \tag{3}$$

Let $s = Nt$ so $ds = Ndt$ and when $t = \pi$ then $s = N\pi$ and $t = -\pi$ then $s = -N\pi$. Hence

$$I = \frac{1}{N}\int_{-N\pi}^{N\pi} \sin t dt \tag{4}$$

and then, if N is an integer, periodicity gives

$$I = \int_{-\pi}^{\pi} \sin t dt = -\cos\pi + \cos\pi = 0 \tag{5}$$

The exception is $n = m$ in which case the first term is just zero.

2. (2) Show by checking whether $f(t) = -f(-t)$ for odd, $f(t) = f(-t)$ for even and neither for neither which of the following are odd, even or neither: $\sin t$, t^3+t, t^3+2t^2 and $|t|$.

Solution:

$$\begin{aligned} \sin{(-t)} &= -\sin t \\ (-t)^3 + (-t) &= -(t^3+t) \\ (-t)^3 + 2(-t)^2 &= -t^3 + t^2 \\ |-t| &= |t| \end{aligned} \tag{6}$$

so odd, odd, neither, even, respectively.

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3. (4) Find the Fourier series representation of the sawtooth function f defined by $f(t) = t$ for $-\pi < t \le \pi$ and $f(t+2\pi) = f(t)$.

Solution: f is odd so $a_n = 0$ for all n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\ t\ \sin nt = -\left.\frac{t\cos nt}{n\pi}\right|_{\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos nt}{n}.$$

The integral on the RHS is zero since it is just a cosine integrated over a full period (or n periods). Thus $b_n = -2\cos(n\pi)/n = -2(-1)^n/n$ which gives

$$f(t) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nt.$$