MA22S3 Tutorial Sheet 5.[1][2]

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Useful facts:

- A function with period 2π has the **Fourier series expansion**

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(int).$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\exp(-int)dx$$

- Parceval's formula:

$$\begin{aligned}\frac{1}{l}\int_{-l/2}^{l/2}|f(t)|^2 dt &= \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\\ &= \sum_{n=-\infty}^{\infty}|c_n|^2\end{aligned}$$

- The **Fourier integral** or Fourier transform:

$$\begin{aligned}f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt}\\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{-ikt}\end{aligned}$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

and so

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

and $\delta(t) = 0$ for $t \neq 0$.

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. (2) Revise Fourier series and Parceval's Theorem. Consider the Fourier expansion of $f(t) = t, \quad -\pi < t < \pi$ with $f(t + 2\pi) = f(t)$ and use the result to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. (2) Show that the Fourier transform of an even function is even, that is, if $f(-t) = f(t)$ then $\widetilde{f(-k)} = \widetilde{f(k)}$.

3. (2) Use integration by parts to show that

$$\int_{-infty}^{\infty} f(t) \frac{d}{dt} \delta(t) dt = -\dot{f}(0) \tag{1}$$

where we are using the notation

$$\frac{d}{dt} f(t) = f(\dot{t}) \tag{2}$$

4. (2) By substituting $y = Ae^{rt}$ and solving for r obtain the solution to $\dot{y} - 3y = 0$, if $y(0) = 1$ calculate the value of A.