MA22S3 Outline Solutions for Tutorial Sheet 9.[1][2]

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Questions

1. (2) Solve $t^2\ddot{y} + 4t\dot{y} + 2y = 0$.

   Solution: The standard substitution $x = e^z$ gives

   $$\frac{d^2y}{d^2z} + (4-1)\frac{dy}{dz} + 2y = 0. \tag{1}$$

   Auxiliary equation is $\lambda^2 + 3\lambda + 2 = 0$ with roots which has roots $\lambda_1 = -2$ and $\lambda_2 = -1$:

   $$y = C_1 e^{-2z} + C_2 e^{-z} = C_1 t^{-2} + C_2 t^{-1} \tag{2}$$

2. (6) Assuming the solution of

   $$(1-t)\dot{y} + y = 0 \tag{3}$$

   has a series expansion about $t = 0$ work out the recursion relation. Write out the first few terms and notice that the series $a_2 = 0$ so the series actually terminates to give $y = a_0(1-t)$. What is the solution with $y(0) = 2$.

   Solution: Well we begin by writing

   $$y = \sum_{n=0}^{\infty} a_n t^n \tag{4}$$

   and so by differentiation we get

   $$\dot{y} = \sum_{n=0}^{\infty} a_n n t^{n-1} \tag{5}$$

   and hence

   $$t\dot{y} = \sum_{n=0}^{\infty} a_n n t^n. \tag{6}$$

   Thus, substituting the differential equation we get

   $$\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \tag{7}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2] Including material from Chris Ford, to whom many thanks.

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \mathrm{stuff}_n t^n \tag{8}$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for $\dot{y}$, (5), to get

$$\dot{y} = \sum_{m=0}^{\infty} a_{m+1}(m+1)t^m. \tag{9}$$

In fact, this looks at first like it gives

$$\dot{y} = \sum_{m=-1}^{\infty} a_{m+1}(m+1)t^m \tag{10}$$

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n, don't get confused, this isn't the original n, we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)t^n - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \tag{11}$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1) + (1-n)a_n]t^n = 0. \tag{12}$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \tag{13}$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0. \tag{14}$$

For $n = 1$ we get

$$a_2 = 0 \tag{15}$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-t)$ for arbitrary a_0. If $y(0) = 2$ then $a_0 = 2$ and $y = 2(1-t)$.