MA22S3 Outline Solutions for Tutorial Sheet 6.[1][2]

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Questions

1. (2) Obtain the solution to $\dot{y} - 3y = e^{-t}$ with $y(0) = 1$; since an initial condition is chosen at $t = 0$, choose $a = 0$.

   Solution: To plug it into the equation, we have $p = -3$ so $I(t) = int_a^t p(t)dt = -3t + 3a = -3t$ if a is set at $a = 0$ hence, from the formula,

   $$y(t) = y(0)e^{3t} + e^{3t}\int_0^t e^{-\tau}e{-}3\tau d\tau = e^{3t} - \frac{1}{4}\left(e^{-t} - e^{3t}\right) = \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t} \tag{1}$$

   Alternatively, rewrite as

   $$e^{-3t}y' - 3ye^{-3t} = e^{-4t}$$

   or

   $$\left(e^{-3t}y\right)' = e^{-4t}$$

   and then integrate.

2. (3) Obtain the solution to $\dot{y} + y\cot t = \cos t$ with $y(\pi/2) = 0$.

   Solution: The quickest thing to do is multiply across by the sine

   $$\sin t\dot{y} + costy = \sin t\cos t \tag{2}$$

   and rewritting

   $$\frac{d}{dt}(\sin ty) = \frac{1}{2}\frac{d}{dt}(\sin^2 t) \tag{3}$$

   hence

   $$\sin ty = \frac{1}{2}\sin^2 t + C \tag{4}$$

   or

   $$y = \frac{1}{2}\sin t + C\text{cosec}\, t \tag{5}$$

   and puting in $t = \pi/2$ we get $0 = 1/2 + C$ or $C = -1/2$. If you substitute that back in you get

   $$y = \frac{1}{2}\sin t - \frac{1}{2}\frac{1}{\sin t} = \frac{1}{2}\frac{\sin^2 t - 1}{\sin t} = -\frac{1}{2}\frac{\cos^2 t}{\sin t} \tag{6}$$

   Of course, it would also be possible to rewrite the integral as

   $$\frac{d}{dt}(\sin ty) = -\frac{1}{2}\frac{d}{dt}(\cos^2 t) \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

[2] Including material from Chris Ford, to whom many thanks.

giving

$$\sin t y = -\frac{1}{2}\cos^2 t + C \tag{8}$$

or

$$y = -\frac{1}{2}\frac{\cos^2 t}{\sin t} + \frac{C}{\sin t} \tag{9}$$

which might, at first, look like a different answer; it isn't though and for the particular initial condition here you get $C = 0$ and hence the same answer again. Yet another way to proceed would be to write

$$\sin t y = \int \sin t \cos t dt = \frac{1}{2}\int \sin 2t dt = -\frac{1}{4}\cos 2t + C \tag{10}$$

and therefore

$$\sin t y = -\frac{1}{4}(1 - 2\sin^2 t) + C = \frac{1}{2}\sin^2 t + C - \frac{1}{4} \tag{11}$$

which is the same as before, up to a redefinition of the arbitrary constant.

Using the formula is harder,

$$I(t) = \int_a^t \cot \tau d\tau = \log|\sin t| - \ln|\sin \pi/2| = \ln|\sin t| \tag{12}$$

You might worry about when you can remove the absolute value sign from sine, but it turns out that isn't a problem. The other integral is

$$\int_a^t e^{I(\tau)} f(\tau) d\tau = \int_{\pi/2}^t \sin\tau\cos\tau dt \tag{13}$$

which you can do using a $u = \cos\tau$ substitution.

3. (3) Obtain a general solution to $(t+1)\dot{y} + y = (t+1)^2$

   Solution: $(t+1)\dot{y} + y = (t+1)^2$ can again be rewritten

$$\frac{d}{dt}[(t+1)y] = t^2 + 2t + 1 \tag{14}$$

so

$$(t+1)y = \frac{1}{3}t^3 + t^2 + t + C \tag{15}$$

or

$$y = \frac{t^3 + 3t^2 + 3t + 1}{3(t+1)} + \frac{C}{t+1} = \frac{1}{3}(t+1)^2 + \frac{C}{t+1} \tag{16}$$

with a redefinition of C to get the nice devision at the end, another way to do this would have been to change variables to $z = t + 1$ at the start.

On the other hand dividing across by $1+t$ means we have to consider $p(t) = 1/(1+t)$; we can go down this route, it is just a question of putting all the constants together as C at the end.

$$\int_a^t \frac{1}{1+\tau} d\tau = \ln(1+t) - \ln(1+a) \tag{17}$$

so lets choose $a = 0$ for convenience. Now $\exp \ln(1+t) = 1+t$ and, noting that $f = (1+t)$ since we divide across by $(1+t)$ to get the equation in to standard form

$$\int_0^t e^{I(\tau)} f(\tau) d\tau = \int_0^t (1+\tau)^2 d\tau = \frac{1}{3}(1+t)^3 - \frac{1}{3} \tag{18}$$

so substituting in to the equation

$$y(t) = \frac{y(0)}{1+t} + \frac{1}{3}(1+t)^3 - \frac{1}{3(1+t)} \tag{19}$$

and so we get the same answer if we define $C = y(0) - 1/3$.