MA22S3 Outline Solution Tutorial Sheet 5.[1,2]

16 November 2009

Questions

1. (4) Express the following function as a Fourier integral:

$$f(x) = \begin{cases} \cos t & |t| < \frac{\pi}{2} \\ 0 & |t| > \frac{\pi}{2} \end{cases}$$

One way to do the required integral is to split the cosine into exponentials.

Solution: (a) Writing f as a Fourier integral $f(t) = \int_{-\infty}^{\infty} e^{ikt}\, \tilde{f}(k)\, dk$. We require the Fourier transform, the important thing is to note that the function is zero outside of $(-\pi/2, \pi/2)$ so the integral outside of that range doesn't contribute to the answer

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{-ikt}\ f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt\ e^{-ikt}\ \frac{e^{it}+e^{-it}}{2} \\
&= \frac{1}{4\pi}\left(\frac{e^{i(1-k)t}}{i(1-k)} + \frac{e^{i(-1-k)t}}{i(-1-k)}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-ik\pi/2}+ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2}-ie^{ik\pi/2}}{i(-1-k)}\right] \\
&= \frac{1}{4\pi}\, 2\cos\left(\frac{k\pi}{2}\right)\ \left(\frac{1}{1-k}+\frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right)\ \frac{1}{1-k^2}.
\end{aligned}$$

Therefore

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\ \cos\left(\frac{k\pi}{2}\right)\ \frac{e^{ikt}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$.

2. (4) Do the following integrals

(a)

$$\int_{-\infty}^{\infty} \delta(t)(t^2+3t+5)dt \tag{1}$$

Solution: The argument of the delta function is zero for $t = 0$ and that's included in the range of integration so we just evaluate the integrand at $t = 0$.

$$\int_{-\infty}^{\infty} \delta(t)(t^2+3t+5)dt = 5 \tag{2}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3
[2] Including material from Chris Ford, to whom many thanks.

(b)

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt \tag{3}$$

Solution: Here the arguement of the delta function is zero when $t = \pi/4$ and this is in the range of integration, so

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{4}$$

(c)

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt \tag{5}$$

Solution: Here, we substitute $t = 1$ into the integrand to give

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt = \ln 1 = 0 \tag{6}$$

(d)

$$\int_{-3}^{3} \delta(t - 4)(t^2 + 3t + 5) dt \tag{7}$$

Solution: The zero of the argument of the delta function is $t = 4$; the delta function is treated as being equal to zero away from zero, so, since $t = 4$ is not in $(-3, 3)$, the range of integration, we get zero.

$$\int_{-3}^{3} \delta(t - 4)(t^2 + 3t + 5) dt = 0 \tag{8}$$