MA22S3 Outline Solutions Tutorial Sheet 4.[1][2]

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Questions

1. (4) Express the following periodic function ($L = 2\pi$) as complex Fourier series

$$f(t) = \begin{cases} 0 & -\pi < t < -a \\ 1 & -a < t < a \\ 0 & a < t < \pi \end{cases} \quad (1)$$

where $a \in (0, \pi)$ is a constant.

Solution: $f(x) = \sum_{n \in Z} c_n e^{int}$ with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt\ e^{-int} f(t) = \frac{1}{2\pi} \int_{-a}^{a} dt\ e^{-int}$$

so that $c_0 = a/\pi$ and

$$c_n = \frac{1}{2\pi} \frac{e^{-int}}{-in}\Big|_{-a}^{a} = \frac{1}{\pi n} \frac{e^{ian} - e^{-ian}}{2i} = \frac{1}{\pi n} \sin an.$$

2. (4) Compute the Fourier transform of $f(t) = e^{-a|t|}$ where a is a positive constant.

Solution: The idea here is to plug $f(t)$ directly in to the equation for $\tilde{f}(k)$ and then use $|t| = t$ for $t > 0$ and $|t| = -t$ for $t < 0$.

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\ e^{-ikt}\ f(t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\ e^{-ikt}\ e^{-a|t|} = \frac{1}{2\pi} \left[\int_{0}^{\infty} dt\ e^{-ikt-at} + \int_{-\infty}^{0} dt\ e^{-ikt+at} \right] \\ &= \frac{1}{2\pi} \left[-\frac{e^{-t(a+ik)}}{a+ik}\Big|_{0}^{\infty} - \frac{e^{t(a-ik)}}{a-ik}\Big|_{-\infty}^{0} \right] \\ &= \frac{1}{2\pi} \left[\frac{1}{a+ik} + \frac{1}{a-ik} \right] = \frac{1}{\pi} \frac{a}{a^2+k^2}. \end{aligned}$$

f can be represented as a Fourier integral

$$f(t) = \int_{-\infty}^{\infty} dk\ e^{ikt}\ \tilde{f}(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk\ \frac{e^{ikt}}{a^2+k^2}.$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3

[2] Including material from Chris Ford, to whom many thanks.