MA22S3 Outline Solutions Tutorial Sheet 3.[1][2]

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Questions

1. (4) $f(t) = \cos t$ for $-\pi/2 < t < \pi/2$ and zero for $-\pi < t < -\pi/2$ and $\pi > t > \pi/2$. It is periodic with period 2π. What is the Fourier series.

   Solution: So this is an even function and hence there is no need to work out the b_n. For a_0 integration gives

   $$a_0 = \frac{1}{\pi}\int_{-\pi}\pi f(t)dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos t dt = \frac{2}{\pi} \tag{1}$$

   for the a_n

   $$a_n = \frac{1}{\pi}\int_{-\pi}\pi f(t)\cos nt dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos t\cos nt dt \tag{2}$$

   because $f(t)$ is zero for t between $-\pi$ and $-\pi/2$, and between $\pi/2$ and π. Now, we do the usual trick

   $$\cos t\cos nt = \frac{1}{2}\left(\cos{(n+1)t} + \cos{(n-1)t}\right) \tag{3}$$

   Hence, integrating

   $$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos t\cos nt dt = \frac{1}{2\pi}\left[\frac{\sin(n+1)t}{n+1} + \frac{cos(n-1)t}{n-1}\right]_{-\pi/2}^{\pi/2} \tag{4}$$

   Next, we can use the usual trignometric identities, or the graph, to see

   $$\begin{aligned} \sin\frac{(n+1)\pi}{2} &= \cos n\pi/2 \\ \sin\frac{(n-1)\pi}{2} &= -\cos n\pi/2 \end{aligned} \tag{5}$$

   to get

   $$a_n = -\frac{1}{\pi}\frac{\cos n\pi/2}{n^2-1} \tag{6}$$

   Now, by examining a graph, it is easy to see $\cos n\pi/2$ is zero for n odd, if n is even it is equal to $(-1)^{n/2}$

   $$f(t) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n \text{ even}}\frac{(-1)^{n/2}}{n^2-1}\cos nt \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

[2] Including material from Chris Ford, to whom many thanks.

2. (2) What is the period of $\sin^2 t$ and $\sin^3 t$.

   Solution: Since $\sin t$ has period 2π, so does $\sin^3 t$, however $\sin^2 t$ has period π since it $(-\sin t)^2 = \sin^2 t$.

3. (2) Find the Fourier series for $\sin^3 t$; a quick way to do this is to regard it as a trigonometry problem, rather than a Fourier series problem, that is use the trigonometric identities to express it in terms of sines and cosines, rather than doing all the integrals: so start by writing $\sin^3 t = \sin^2 t \sin t$ and then write $\sin^2 t$ in terms of $\cos 2t$.

   Solution: Using trig identities

$$\sin^3 t = \sin^2 t \sin t = \frac{1}{2}(1 - \cos 2t) \sin t \tag{8}$$

   and again use the trig identity for $\cos 2t \sin t$ to give

$$\sin^3 t = -\frac{1}{4} \sin 3t + \frac{3}{4} \sin t \tag{9}$$

   In other words, $b_1 = 3/4$ and $b_3 = -1/4$ and all the others are zero.