MA22S3 Outline Solutions Tutorial Sheet 2.[1]

14 October 2009

1. (4) Find the Fourier series representation of the sawtooth function f defined by $f(t) = t$ for $-\pi < t < \pi$ and $f(t + 2\pi) = f(t)$.

   Solution: f is odd so $a_n = 0$ for all n.

   $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\ t\ \sin nt = -\left.\frac{t\cos nt}{n\pi}\right|_{\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nt}{n}.$$

   The integral on the RHS is zero since it is just a cosine integrated over a full period (or n periods). Thus $b_n = -2\cos(n\pi)/n = -2(-1)^n/n$ which gives

   $$f(t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nt.$$

2. (2) What is $f(\pi)$? If the answer to question one is

   $$f(t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nt. \tag{1}$$

   what value does the Fourier series give at $t = \pi$?

   Solution: Well $f(\pi) = \pi$ from the definition, but since all the sines give zero, the Fourier series gives zero.

3. (2) By considering $t = \pi/2$ derive a formula for π.

   Solution: Subbing in $t = \pi/2$ gives

   $$f(\pi/2) = \frac{\pi}{2} = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi/2. \tag{2}$$

   Now, if you draw a graph of $\sin n\pi/2$ it is clear that it is zero for n even and for n odd, let $n = 2m + 1$ and

   $$\sin\frac{(2m+1)\pi}{2} = (-1)^m \tag{3}$$

   Hence

   $$\pi = -4\sum_{m=0}^{\infty}\frac{(-1)^{2m+1+m}}{2m+1} = 4\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} \tag{4}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3