MA22S3 Outline Solutions for Tutorial Sheet 10[1][2]

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Questions

1. (3) Use the recursion relation

$$a_{n+2} = \frac{2(n-\alpha)a_n}{(n+1)(n+2)}$$

to obtain polynomial solutions of Hermite's equation $\ddot{y} - 2t\dot{y} + 2\alpha y = 0$ for $\alpha = 3$ and 4.

Solution: So this is just a question of turning the handle on the recursion formula. Now, the important point is that only one of the two series is a polynomial: what you are asked for here is the polynomial solution. A polynomial is a terminating series, it doesn't go off to infinity and the recursion terminates because of the $n - \alpha$ in the numerator, of course, for α even this gives zero for an even n and for α odd, for an odd n. Thus, for $\alpha = 3$ it is the odd series that terminates, so we start with $a_0 = 0$ to get the odd series, a_1 is arbitrary and $n = 1$ gives

$$a_3 = \frac{2(1-3)}{(1+1)(1+2)}a_1 = -\frac{2}{3}a_1 \tag{1}$$

and $n = 3$ has $3 - 3 = 0$ in the numerator, so $a_5 = 0$ and the polynomial is

$$y(t) = a_1\left(t - \frac{2}{3}t^3\right) \tag{2}$$

Similarly, for $\alpha = 4$ it is the even series that terminates, so a_0 is arbitrary, $a_1 = 0$ and, using $n = 0$

$$a_2 = \frac{-8}{2}a_0 = -4a_0 \tag{3}$$

and, using $n = 2$

$$a_4 = -\frac{4}{12}a_2 = \frac{4}{3}a_0 \tag{4}$$

Finally, $n = 4$ has zero in the numerator, giving $a_6 = 0$ and

$$y(t) = a_0\left(1 - 4t^2 + \frac{4}{3}t^4\right) \tag{5}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

2. (5) Use the method of Frobenius to obtain the general solution to the ODE

$$t\ddot{y} + 2\dot{y} + ty = 0.$$

Solution: So, since we are told to use the method of Froebenius, we substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{6}$$

Even if you weren't told this was a method of Froebenius problem, you would soon find that the ordinary method doesn't give two solutions. Alternatively, you could notice that if you write the equation so nothing multiplies y'' you have coefficients with singularities, that is in this case, the $2/x$ multiplying y'.

Now, substituting into the equation gives

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r)]a_n t^{n+r-1} + \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0. \tag{7}$$

so, moving the first power up to the second one, this gives

$$\sum_{n=-2}^{\infty} [(n+2+r)(n+r+1) + 2(n+r+2)]a_{n+2} t^{n+r+1} + \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0 \tag{8}$$

or, taking the first two terms out

$$\begin{aligned} r(r+1)a_0 t^{r-1} + (r+1)(r+2)a_1 t^r &+ \\ \sum_{n=0}^{\infty} [(n+2+r)(n+r+3)]a_{n+2} t^{n+r+1} &+ \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0. \end{aligned} \tag{9}$$

So, if $r = 0$ or $r = -1$ then there is no constraint on a_0. Notice that $r = -1$ allows two solutions because, if $r = -1$ there is no equation for either a_0 or a_1. For $r = -1$ the recursion is

$$a_{n+2} = \frac{1}{(n+1)(n+2)} \tag{10}$$

so the first few non-zero terms are

$$y = \frac{1}{t}\left[a_0\left(1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots\right) + a_1\left(t + \frac{1}{6}t^3 \dots\right)\right] \tag{11}$$

For $r = 0$ the recursion is

$$a_{n+2} = \frac{1}{(n+2)(n+3)} \tag{12}$$

and $a_1 = 0$, this means that the $r = 0$ solution is

$$y = a_0\left(1 + \frac{1}{6}t^2 + \dots\right) \tag{13}$$

Notice that the $r = 0$ solution is actually just the a_1 solution for $r = -1$. This is just as well because there would be too many solutions otherwise: the recommended approach is to take both a_0 solutions. Notice the subtle way the method of Froebenius problems often work out. There is quite a lot to this subject we have only touched on.