MA22S3 Tutorial Sheet 1. Outline solns.[1]

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Questions

1. (2) Establish that

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \cos nt = 0 \tag{1}$$

for integers n and m.

Solution: This is an exercise in trigonometry and uses first the trignometric identity

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \cos nt = \frac{1}{2}\int_{-\pi}^{\pi} dt\ [\sin{(n-m)t} - \sin{(n+m)t})] = 0 \tag{2}$$

and then the fact that the integral of a trignometric function over its whole period is zero. This fact is used a lot; take sine for example, and let N be an integer. Consider

$$I = \int_{-\pi}^{\pi} \sin Ntdt \tag{3}$$

Let $s = Nt$ so $ds = Ndt$ and when $t = \pi$ then $s = N\pi$ and $t = -\pi$ then $s = -N\pi$. Hence

$$I = \frac{1}{N}\int_{-N\pi}^{N\pi} \sin tdt \tag{4}$$

and then, if N is an integer, periodicity gives

$$I = \int_{-\pi}^{\pi} \sin tdt = -\cos\pi + \cos\pi = 0 \tag{5}$$

The exception is $n = m$ in which case the first term is just zero.

2. (2) Establish that

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \sin nt = 0, \tag{6}$$

if $m \neq n$ (both m and n are positive integers). What happens when $m = n$? By the way, the same story holds for $\int_{-\pi}^{\pi} dx\ \cos mt\ \cos nt$ but you need not do that one for this problem sheet.

Solution: This is almost the same

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \sin nt = \frac{1}{2}\int_{-\pi}^{\pi} dt\ [\cos{(n-m)t} - \cos{(n+m)t})] = 0 \tag{7}$$

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and then the fact that the integral of a trignometric function over its whole period is zero. The exception is $n = m$ in which case the first term is one and so we get

$$\int_{-\pi}^{\pi} dt\ \sin mt\ \sin nt = \frac{1}{2}\int_{-\pi}^{\pi} dt\ = \pi \tag{8}$$

3. (2) Show by checking whether $f(t) = -f(-t)$ for odd, $f(t) = f(-t)$ for even and neither for neither which of the following are odd, even or neither: $\sin t$, t^3+t, t^3+2t^2 and $|t|$.

   Solution:

$$\begin{aligned} \sin(-t) &= -\sin t \\ (-t)^3 + (-t) &= -(t^3 + t) \\ (-t)^3 + 2(-t)^2 &= -t^3 + t^2 \\ |-t| &= |t| \end{aligned} \tag{9}$$

   so odd, odd, neither, even, respectively.

4. (2) What is the period of $\sin(2\pi t/L)$ and $\cos(2\pi t/L)$?

   Solution:

$$\sin\frac{2\pi(t+L)}{L} = \sin\left(\frac{2\pi t}{L} + 2\pi\right) = \sin\frac{2\pi t}{L} \tag{10}$$

   and any smaller number would not have worked, the cosine works the same way. Or put another way, if

$$f(t) = \sin\frac{2\pi(t)}{L} \tag{11}$$

   then

$$f(t+T) = \sin\frac{2\pi(t+T)}{L} = \sin\left(\frac{2\pi t}{L} + \frac{2\pi T}{L}\right) \tag{12}$$

   and since $\sin(\theta + 2\pi) = \sin\theta$, we have $f(t+T) = f(t)$ provided $T = L$.