MA22S3: Vector space: inner products and projections.[1]

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Euclidean vector spaces have another important structure beyond the linear structure captured by the definition of a vector space: this is the dot product or scalar product. For two Euclidean vectors $\mathbf{u}$ and $\mathbf{v}$ the dot product is

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \tag{1}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We know that if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{2}$$

The definition of the inner product generalizes this structure. Since the dot in dot product is specific to Euclidean vectors, a new notation is needed for the general case, several are used, here we will use the Dirac notation.

- For a real vector space V an *inner product* is a map from pairs of vectors to real numbers

$$\begin{array}{rcl} \langle\ |\ \rangle : V\times V & \to & \mathbf{R} \\ (u,v) & \mapsto & \langle u|v\rangle \end{array} \tag{3}$$

  satisfying,

  1. Symmetry: $\langle u|v\rangle = \langle v|u\rangle$.
  2. Bilinearity: $\langle \lambda_1 u_1 + \lambda_2 u_2|v\rangle = \lambda_1\langle u_1|v\rangle + \lambda_2\langle u_2|v\rangle$ for u_1, u_2 and v in V and λ_1 and λ_2 in $\mathbf{R}$ and $\langle u|\lambda_1 v_1 + \lambda_2 v_2\rangle = \lambda_1\langle u|v_1\rangle + \lambda_2\langle u|v_2\rangle$ for u, v_1 and v_2 in V and λ_1 and λ_2 in $\mathbf{R}$.
  3. Positive-definiteness: $\langle u|u\rangle \geq 0$ with $\langle u|u\rangle = 0$ only when $u = 0$.

  A *real inner product space* is a real vector space with an inner product.

It is easy to check in the case of the component-wise definition of the dot-product that it is an inner product. It is harder to check with the non-component definition, $|\mathbf{u}||\mathbf{v}|\cos\theta$, but can be done. The third condition is important, it expresses what makes the dot product different from general bilinear symmetric maps: the dot product of a vector with itself is the length squared: $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$. More generally the notation

$$\|u\| = \sqrt{\langle u|u\rangle} \tag{4}$$

and $\|u\|$ is a *norm*.

The dot product also allow us to say when two vectors are at right angles, for non-zero vectors $\mathbf{u}\cdot\mathbf{v} = 0$ only if θ, the included angle, is $\pi/2$. This generalizes to inner product spaces.

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- Two vectors u and v in an inner product space are *orthogonal* if $\langle u|v\rangle = 0$.
- A set of vectors $\{u_1, u_2, \ldots, u_r\}$ is called orthogonal if every pair of vectors in it orthogonal: $\langle u_i|u_j\rangle = 0$ for each $i \neq j$. The set is *orthonormal* if it is orthogonal and all the vectors have norm one: $\langle u_i|u_i\rangle = 1$ for all i. This can be written as $\langle u_i|u_j\rangle = \delta_{ij}$ where we have used the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{5}$$

Now, in $\mathbf{R}$ the usual orthonormal basis is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, but this isn't unique, rotations give other sets, so, for exaple, $\{(\mathbf{i}+\mathbf{j})/\sqrt{2}, (\mathbf{i}-\mathbf{j})/\sqrt{2}, \mathbf{k}\}$ is also orthonormal. In a finite-dimensional vector spaces any basis set can be used to construct an orthonormal basis. This can be done using an algorithm called *Gram-Schmidt orthonormalization*: in short every finite-dimensional vector space has an orthonormal basis.

One important thing about an orthonormal vector space is that you can use *orthogonal projection* to work out components. For example, in $\mathbf{R}^3$ we know $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a orthonormal basis set, so given a vector $\mathbf{v}$ we can write

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \tag{6}$$

for some v_1, v_2 and v_3; the question is what values do v_1, v_2 and v_3 take? To work this out we dot-product both sides by $\mathbf{i}$ to get

$$\mathbf{v}\cdot\mathbf{i} = v_1\mathbf{i}\cdot\mathbf{i} + v_2\mathbf{j}\cdot\mathbf{i} + v_3\mathbf{k}\cdot\mathbf{i} \tag{7}$$

Now, since $\mathbf{i}\cdot\mathbf{i} = 1$ and $\mathbf{j}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{i} = 0$ we get

$$v_1 = \mathbf{v}\cdot\mathbf{i} \tag{8}$$

Dotting across by the other two basis vectors in a similar way gives $v_2 = \mathbf{v}\cdot\mathbf{j}$ and $v_3 = \mathbf{v}\cdot\mathbf{k}$. Geometrically the dot-product corresponds to projection and so this process shows that each of the components is equal to the projection of the vector onto the relevant direction.

It is useful to study this process in a slightly more general way: let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in n-dimensional Euclidean space. If $\mathbf{v}$ is a vector in this space

$$\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{e}_i \tag{9}$$

Now, to calculate the a_i's we dot across by one of the basis vectors, say $\mathbf{e}_j$:

$$\mathbf{v}\cdot\mathbf{e}_j = \sum_{i=1}^{n} a_i\mathbf{e}_i\cdot\mathbf{e}_j \tag{10}$$

Next, using $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ which follows from the orthonormality of the basis set,

$$\mathbf{v} \cdot \mathbf{e}_j = \sum_{i=1}^{n} a_i \delta_{ij} = a_j \tag{11}$$

so

$$\mathbf{v} = \sum_{i=1}^{n} (\mathbf{e}_i \cdot \mathbf{v}) \mathbf{e}_i \tag{12}$$

So far, the only example of an inner product we have looked at is the dot product. Obviously we are interested in an inner product for periodic functions and it is easy to check that for the space of functions of period L, with some integrability condition, for two such functions, $f(t)$ and $g(t)$,

$$\langle f|g\rangle = \int_{-L/2}^{L/2} fg dt \tag{13}$$

is an inner product: again there are technical restrictions, positive definiteness requires functions requires some sort of saneness requirement, requiring that the functions have at most a finite number of discontinuities works. There is also a set of orthogonal vectors for these functions provided by the sines and cosines

$$\{1/2, \cos 2\pi t/L, \cos 4\pi t/L, \cos 6\pi t/L, \ldots, \sin 2\pi t/L, \sin 2\pi t/L, \sin 4\pi t/L, \sin 6\pi t/L, \ldots\}$$

Integrating shows that

$$\begin{aligned}
\int_{-L/2}^{L/2} \cos\frac{2\pi nt}{L} \cos\frac{2\pi mt}{L} &= \frac{L}{2}\delta_{nm} \\
\int_{-L/2}^{L/2} \sin\frac{2\pi nt}{L} \sin\frac{2\pi mt}{L} &= \frac{L}{2}\delta_{nm} \\
\int_{-L/2}^{L/2} \cos\frac{2\pi nt}{L} \sin\frac{2\pi mt}{L} &= 0
\end{aligned} \tag{14}$$

for positive integers m and n; we will see that this orthonormal set gives us the Fourier series.