MA22S3: Vector space: definition and bases.[1]

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The vector spaces we are most familiar with are Euclidean vector space like $\mathbf{R}^3$; the set of three-dimensional vectors. Our goal here is define abstractly the important structure of these Euclidean vector space so that some of the important results from the theory of Euclidean vector spaces can be applied in other examples, such as the example that will be important to us here: periodic functions.

A vector space is a set with addition and scalar multiplication; the definition explains what we mean by addition and scalar multiplication: it lists all the things that are important about these operations. We also have to decide what scalars we have in mind when we say scalar multiplications, for Euclidean vectors, the scalar is a real number, however, that is an unnecessary restriction, it is possible to define vector spaces where the scalars are complex numbers for example. It turns out that the important restriction is that the scalars form a mathematical structure called a *field*; we won't go into that here, suffice to say that the real numbers $\mathbf{R}$ and the complex numbers $\mathbf{C}$ are both examples of fields. Thus, we will give the definition of a vector space with a general field $\mathbf{F}$, but, in practice we will only look at real and complex vector spaces where $\mathbf{F} = \mathbf{R}$ or, less commonly, $\mathbf{F} = \mathbf{C}$.

- A *vector space* over a field $\mathbf{F}$ is a set V with an addition operation:

  $$u + v \in V \tag{1}$$

  for all u, v in V and a scalar multiplication,

  $$\lambda u \in V \tag{2}$$

  for $u \in V$ and $\lambda \in \mathbf{F}$. For all u, v and w in V and λ and μ in $\mathbf{F}$, these operations are required to satisfy

  1. Symmetry: $u + v = v + u$.
  2. Associativity: $(u + v) + w = u + (v + w)$.
  3. Two distributive laws: $\lambda(u + v) = \lambda u + \lambda v$ and $(\lambda + \mu)u = \lambda u + \mu u$.
  4. Identity: there exists an element 0 such that $v + 0 = v$.
  5. Inverse: there exists an element $-v$ such that $v + (-v) = 0$, of course, we write $u - v$ for $u + (-v)$.
  6. Compatibility: $(\lambda\mu)u = \lambda(\mu u)$.
  7. Identity for scalar multiplication: if 1 is the identity in $\mathbf{F}$; $1u = u$.

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The numbered niceness conditions just list the important properties that addition and scalar multiplication have for Euclidean vectors; ensuring that a general vector space has these same properties. Notice that it is important that not only is there an addition operation, but that adding two vectors gives another vector in the set: we say that a vector space is *closed* under addition. It is also closed under scalar multiplication.

It is easy to check that the space of, for example three-dimensional, Euclidean vectors gives a vector space, for example, if $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^3$ so is $\mathbf{u}+\mathbf{v}$ and, $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. It is this sort of finite-dimensional vector space that motivated the definition of a vector space and which will provide the intuition we will need to deal with more complicated examples.

Another, maybe more surprising example, is given by periodic functions.

- A function $f(t)$ is *periodic* if there exists an L such that for all t $f(t + L) = f(t)$. The smallest such an L is called the *period*.

Consider the set of all functions with period L. Now if $f(t)$ and $g(t)$ have periods L then so does $h(t) = f(t) + g(t)$:

$$h(t + L) = f(t + L) + g(t + L) = f(t) + g(t) = h(t) \tag{3}$$

Similarly, if $h(t) = af(t)$ for $a \in \mathbf{R}$ then $h(t + L) = af(t + L) = af(t) = h(t)$ so this space has closed addition and scalar multiplication operations. It would also be easy to check that these operations also satisfy all the niceness conditions.

Thus, the space of functions of period L is another example of a vector space: we will see that it differs in an important way from examples like $\mathbf{R}^3$ it is infinite dimensional. To see what this means we need to first of all define dimension and to that we need to discuss linear independence.

- For a vector space V a set of non-zero vectors $\{u_1, uthatcanbeused_2, \ldots, u_n\}$ is *linearly independent* if the only way scalars λ_1, λ_2 up to λ_n give

  $$\lambda_1 u_1 + \lambda_2 u_2 \ldots \lambda_n u_n = 0 \tag{4}$$

  is $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

In other words a set is linearly independent if these are no non-trivial linear combinations of the vectors that give zero. Now, in $\mathbf{R}^3$ the basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are linearly independent, but, say $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (2, 2, 1)$ and $\mathbf{w} = (0, 0, -1)$ are not linearly independent, because,

$$that\mathbf{v} - 2\mathbf{u} + \mathbf{w} = 0 \tag{5}$$

Now, for many vector spaces there is a limit on how many vectors you can have in a linearly independent set: take three-dimensional Euclidean space, $\mathbf{R}^3$ as an example and take $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as the linearly independent set. Now, say we have another vector $\mathbf{v} = (v_1, v_2, v_3)$, we know $\mathbf{v} - v_1\mathbf{i} - v_2\mathbf{j} - v_3\mathbf{k} = 0$. This limit actually defines the dimension

- If such a set exists, the *dimension* of a vector space is the maximal size of a linearly independent set. Any linearly independent set of size equal the dimension is called a *basis*.

Notice first the caveat *if such a set exists*; not all vector spaces have a finite dimension. The word maximal here means something very similar to maximum, but it explicitly notes that the maximum will be obtain in many instance: if there is a basis set it will not be unique, for example along with $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the set $\{\mathbf{i}+\mathbf{j}, \mathbf{i}-\mathbf{j}, \mathbf{k}\}$ is a basis for $\mathbf{R}^3$.

Obviously, we think of a basis as a set of vectors as a set of vectors over which any other vector can be decomposed. That idea is implicit in the definition above; say $\{u_1, u_2, \ldots, u_n\}$ is a basis for some vector space V and v is any element in V. Now $\{u_1, u_2, \ldots, u_n, v\}$ have more than n elements, but, by the definition of the basis, n is the dimension, so this set must be linearly dependent; there exist non-trivial λ_1 to λ_{n+1} such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n + \lambda_{n+1} v = 0 \tag{6}$$

Moreover, since the set of u_i's is a basis, we know $\lambda_{n+1} \neq 0$; having zere λ_{n+1} would contradict the linear independence of the basis. Now, let $a_i = -\lambda_i/\lambda_{n+1}$ for all i from one to n. Dividing the equation by λ_{n+1} and rearranging a small by gives

$$v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n \tag{7}$$

so the v can be written in terms of the u_i.