MA22S3: Motivating example for Fourier series, solving a linear differential equation.[1]

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One of the problems with introducing Fourier series is that it is not always clear to students what Fourier series are for. Mostly the only way around that is to promise them that Fourier series are important everywhere. Here, one vague motivating example is given: when solving a linear system of differential equation, the problem is untangled by writing everything in terms of eigenvectors.

So, the problem is to find the general solution for the system

$$\begin{aligned} \frac{dx}{dt} &= 3x + y \\ \frac{dy}{dt} &= x + 3y \end{aligned} \tag{1}$$

In some ways this looks easy, the equations are first order and linear; the difficulty is that they are mixed up, the equation for $\dot{x}$ has x's and y's and so does the equation for $\dot{y}$. The way around this is to rewrite everything in terms of eigenvectors of the matrix mixing the two variables. Writing the equation in matrix form

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x} \tag{2}$$

where

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{3}$$

and

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

To calculate eigenvectors and eigenvalues we first solve the characteristic equation

$$\det A - \lambda\mathbf{1} = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 1 = 0 \tag{5}$$

giving $\lambda^2 - 6\lambda + 8 = 0$ with solutions $\lambda_1 = 4$ and $\lambda_2 = 3$. The defining property of an eigenvector is that $\mathbf{e}$ is an eigenvector with eigenvalue λ if and only if

$$A\mathbf{e} = \lambda\mathbf{e} \tag{6}$$

To actually work out eigenvectors we just write substitute a general vector into this equation, using a and b as unknowns say, we have

$$\mathbf{x}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3.html

is a eigenvector with eigenvalue $\lambda_1 = 4$ if

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix} \tag{8}$$

This gives what at first seems to be two equation

$$\begin{aligned} 3a + b &= 4a \\ a + 3b &= 4b \end{aligned} \tag{9}$$

but, in fact, these equations are both the same, the both say $a = b$. This is typical, by the linearity of the equation for $\mathbf{x}_1$, if $\mathbf{x}_1$ is an eigenvector, so is $\mu\mathbf{x}_1$ for any μ. Hence, we shouldn't expect to solve the eigenvalue equation to give a and b; rather, as we see actually happens, the equation should give a in terms of b. Now, rather than the whole line of eigenvectors, we want an example of eigenvector so we need to make a choice of a; this can really be anything, here I am just going to choose $a = 1$ which means $b = 1$ as well, since $a = b$; I could equally well have decided to require $|\mathbf{x}_1| = 1$ which would have given $a = 1/\sqrt{2}$.

The other eigenvector can be worked out in a similar way and the eigenvectors and eigenvalues are $\lambda_1 = 4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

and $\lambda_2 = 2$ with

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{11}$$

Now the trick is to substitute

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f(t)\mathbf{x}_1 + g(t)\mathbf{x}_2 \tag{12}$$

In other words we are expanding $\mathbf{x}$ in terms of the eigenvectors. This amounts to a change of basis, we are going to work out f and g instead of x and y; this is no big change since x and y can be easily calculated from f and g, by the above

$$\begin{aligned} x &= f + g \\ y &= f - g \end{aligned} \tag{13}$$

Now we substitute this expansion back in to the equation

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{14}$$

giving

$$\dot{f}\mathbf{x}_1 + \dot{g}\mathbf{x}_2 = A(f\mathbf{x}_1 + g\mathbf{x}_2) \tag{15}$$

Now, again by linearity of the matrix multiplication

$$\dot{f}\mathbf{x}_1 + \dot{g}\mathbf{x}_2 = fA\mathbf{x}_1 + gA\mathbf{x}_2 = 4f\mathbf{x}_1 + 2\mathbf{x}_2 \tag{16}$$

Now, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent vector

$$
\begin{aligned}
\dot{f} &= 4f \\
\dot{g} &= 2g
\end{aligned}
\tag{17}
$$

which equations can be solved to give

$$
\begin{aligned}
f &= C_1 e^{4t} \\
g &= C_2 e^{2t}
\end{aligned}
\tag{18}
$$

What we have seen is that the equation is solved by untangling it and this is done by expanding in terms of eigenvectors. This is supposed to serve as a motivation for the Fourier series since sine and cosine are *eigenfunctions* of the operator d^2/dt^2.