Note ODE.1[1,2] 19 February 2007 as 231.III.1 and 8 December 2010 in this form.

Part III: ODEs

A **differential equation** is an equation involving derivatives. An **ordinary differential equation (ODE)** is a differential equation involving a function, or functions, of only one variable. If the ODE involves the nth, and lower, derivatives it is said to be an n**th order ODE.** Let y be a function of one variable t, for neatness, we will try to always use t as the dependent variable and dot for derivative. An equation like

$$\dot{y}y + \sin ty = \cos t \tag{1}$$

is a first order ODE, we will see soon that it is not a linear ODE and we will only be looking at linear ODEs, but, the important thing for what we are discussing here is that the highest derivative is the $\dot{y}$.

$$\ddot{y} + \dot{y}y + \sin ty = \cos t \tag{2}$$

is second order. A function satisfying the ODE is called **a solution** of the ODE.

Linear ODEs (2 types)

There are two types of linear ODEs

1. **Homogeneous**: If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where A and B are arbitrary constants.

2. **Inhomogeneous**: If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where $A + B = 1$.

where, obviously, the point is in a homogeneous equation, all the terms are y terms, whereas the inhomogeneous equation has an extra **forcing** term.

- **Homogeneous example:** The equation

  $$\ddot{y} + p(t)\dot{y} + q(t)y = 0 \tag{3}$$

  is homogeneous, where $p(t)$ and $q(t)$ are some, given, functions of t. Now substituting $Ay_1 + By_2$ gives

  $$\frac{d^2}{dt^2}(Ay_1+By_2)+p\frac{d}{dt}(Ay_1+By_2)+q(Ay_1+By_2) = A(\ddot{y}_1+p\dot{y}_1+qy_1)+B(\ddot{y}_2+p\dot{y}_2+qy_2) = 0 \tag{4}$$

  when y_1 and y_2 are solutions.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2]Based partly on lecture notes taken by John Kearney

- **Inhomogeneous example:** The equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = f(t) \tag{5}$$

is homogeneous, where $p(t)$, $q(t)$ and $f(t)$ are some, given, functions of t. Now substituting $Ay_1 + By_2$ gives

$$\frac{d^2}{dt^2}(Ay_1+By_2)+p\frac{d}{dt}(Ay_1+By_2)+q(Ay_1+By_2) = A(\ddot{y}_1+p\dot{y}_1+qy_1)+B(\ddot{y}_2+p\dot{y}_2+qy_2) = (A+B)f \tag{6}$$

when y_1 and y_2 are solutions. Hence $Ay_1 + By_2$ is a solution is $A + B = 1$.

In either case, the main point is that a linear ODE is one where there is some sort of additive structure and, in a linear equation, there are no non-linear functions of y and its derivatives, no y^2 terms, or $\sin y$ terms or $y\dot{y}$ terms or anything like that.

The **general first order linear ODE**, for a single function, can be written

$$a(t)\dot{y}(t) + b(t)y(t) = f(t) \tag{7}$$

where $a(t)$, $b(t)$ and $f(t)$ are arbitrary functions. The equation is homogeneous if $f = 0$. A common standard form is write the equation as

$$\dot{y}(t) + p(t)y(t) = f(t) \tag{8}$$

where $p = b/a$ and f/a has been renamed back to f.

The **general second order linear ODE** is

$$a(t)\ddot{y}(t) + b(t)\dot{y}(t) + c(t)y(t) = f(t) \tag{9}$$

where $a(t)$, $b(t)$, $c(t)$ and $f(t)$ are arbitrary functions and the equation is homogeneous if $f = 0$. Again, another standard form is

$$\ddot{y}(t) + p(t)\dot{y}(t) + q(t)y(t) = f(t) \tag{10}$$

An nth order differential equation will have n independent solutions, so a first order differential equation has one solution, meaning that there is one arbitrary constant, a second order will have two, and so on. Often, initial conditions will fix the arbitrary constants, so a first order equation needs one initial condition, the value of $y(0)$ for example, a second order differential equation needs two, the values of $y(0)$ and $\dot{y}(0)$ for example.

First order linear differential equations.

All solutions of

$$\dot{y}(t) + p(t)y(t) = f(t) \tag{11}$$

can be written

$$y(t) = Cy_1(t) + y_p(t) \tag{12}$$

where $y_1(t)$ is a solution of the **corresponding** homogeneous equation

$$\dot{y}(t) + p(t)y(t) = 0 \tag{13}$$

and $y_p(t)$ is any one solution of the full equation. Hence, once you can find one solution to the problem, you can find the whole set of solutions, with the arbitrary constant, by adding the solution to the corresponding homogeneous equation.

This can be demonstrated in this case by explicit construction. Let

$$\dot{y}(t) + p(t)y(t) = f(t) \tag{14}$$

The trick is to multiply across by a function $\lambda(t)$, called **the integrating factor** so the we can rewrite the left hand side as a product. Hence, we want a λ so that

$$\frac{d}{dt}(\lambda y) = \lambda \dot{y} + \lambda p y \tag{15}$$

Doing the differentiation and matching up, we see that this means

$$\dot{\lambda} = p\lambda \tag{16}$$

or, by integrating

$$lambda(t) = e^{I(t)} \tag{17}$$

where where

$$I(t) = \int_a^t p(z)dz. \tag{18}$$

and a is an arbitrary constant, in fact the choice of a doesn't alter the eventual solution, changes of a are basically absorbed into a redefinition of the arbitrary constant.

Now, after multiplying across by the integrating factor, the differential equation can be rewritten

$$\frac{d}{dt}e^{I(t)}y(t) = e^{I(t)}f(t) \tag{19}$$

Integrate from a to t

$$e^{I(t)}y(t) - e^{I(a)}y(a) = \int_a^t dz e^{I(z)} f(z). \tag{20}$$

with $e^{I(a)} = 1$. This gives

$$y(t) = Cy_1(t) + y_p(t), \tag{21}$$

with $y_1(t) = e^{-I(t)}$, $y_p(t) = e^{-I(t)} \int_a^t e^{I(z)} f(z)dz$ and $C = y(a)$. so

$$y(t) = y(0)e^{-I(t)} + e^{-I(t)} \int_a^t e^{I(z)} f(z)dz \tag{22}$$

In practise, this method will always find a solution, but, often, it is quicker just to stare at the equation and then guess a solution and check it works, or to actually use the integrating factor rather than just plugging stuff into the formula.

- **Example** Find all solutions of the ODE 1

$$\dot{y}(t) + \frac{1}{t}y(t) = t^3. \tag{23}$$

Here $p(t) = 1/t$ which has a non-integrable singularity at $t = 0$! Work with $t > 0$, or $t < 0$. First, the integrating factor $I(t) = \int_a^t p(z)dz = logt$ provided we set $a = 1$. $e^{I(t)} = t$ so that the ODE can be written

$$\frac{d}{dt}(ty) = t^4. \tag{24}$$

Integrating gives $ty = \frac{1}{5}t^5 + C$ or $y = \frac{1}{5}t^4 + C/t$.

Second order case

All solutions, or the **general solution** of

$$\ddot{y}(t) + p(t)\dot{y}(t) + q(t)y(t) = f(t) \tag{25}$$

are given by

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \tag{26}$$

where y_1, y_2 are linearly independent solutions of the **corresponding** homogeneous equation

$$\ddot{y}(t) + p(t)\dot{y}(t) + q(t)y(t) = 0 \tag{27}$$

and $y_p(t)$ is a solution of the full equation. C_1 and C_2 are arbitrary constants. This isn't proved here, but it is easy to understand why it would be the case: this is a second order equation so it nears to arbitrary constant, in the initial value problem, one matches $y(0)$ and the other $\dot{y}(0)$. Now, if you have a solution, adding a solution of the corresponding homogeneous problem gives you another solution and the homogeneous problem also has a two-dimensional space of solutions, so it all matches up. $y_p(t)$ is called a **particular integral**. The general solution is sometimes written

$$y(t) = y_c(t) + y_p(t) \tag{28}$$

where $y_c(t) = C_1 y_1(t) + C_2 y_2(t)$ is called the **complementary function**. It is the general solution of the homogeneous form of the ODE.

Constant Coefficients

We now consider the special case where the coefficients a, b and c are constants

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = f(t). \tag{29}$$

This type of equation has a nice interpretation as a **damped/driven oscillator**. y is the displacement from equilibrium. Recall the equation for a simple harmonic oscillator

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) \tag{30}$$

Now add in a damping force proportional to the velocity dy/dt and a driving force $f(t)$, which may be periodic or non-periodic,

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) - \gamma\frac{dy(t)}{dt} + d(t) \tag{31}$$

which is a linear ODE with constant coefficients.

So, back to the general constant coefficient form, the first step in solving ODEs of this type is to find two solutions of the homogeneous equation

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = 0. \tag{32}$$

This equation has simple exponential solutions of the form $y(t) = e^{\lambda t}$. Differentiating $\dot{y}(t) = \lambda e^{\lambda t}$ and $\ddot{y}(t) = \lambda^2 e^{\lambda t}$ so that

$$a\ddot{y}(t) + b\dot{y} + cy = (a\lambda^2 + b\lambda + c)y \tag{33}$$

which is zero provided

$$a\lambda^2 + b\lambda + c = 0. \tag{34}$$

This is called an **auxiliary equation**. Thus $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ where λ_1 and λ_2 are roots of the quadratic auxiliary equation. The complementary function, if $\lambda_1 \neq \lambda_2$, is $y_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$.

If $\lambda_1 = \lambda_2$ we only have one exponential solution. In this case a second solution of the ODE is $y(t) = te^{\lambda_1 t}$ and $y_c(t) = C_1 e^{\lambda_1 t} + C_2 te^{\lambda_1 t}$. In the oscillator model this special case corresponds to critical damping. This trick is justified by the fact it works; there are ways to derived it, for example, by converting the equation into two first order equations using $y_1 = y$ and $y_2 = y'$ and then diagonalizing the corresponding matrix equation and solving using an integrating factor. In practise, the easiest thing is to keep adding powers of t until you have two solutions.

You might wonder why te^{λ_1} is a solution and why we don't get a solution of this form when there are two distinct solutions to the auxiliary equation. To see this, consider substituting

$$y = te^{\lambda_1 t} \tag{35}$$

into the equation, we have

$$\dot{y} = e^{\lambda_1 t} + \lambda_1 te^{\lambda_1 t} \tag{36}$$

and

$$\ddot{y} = 2\lambda_1 e^{\lambda_1 t} + \lambda_1^2 te^{\lambda_1 t} \tag{37}$$

so the equation becomes

$$a(2\lambda_1 e^{\lambda_1 t} + \lambda_1^2 t e^{\lambda_1 t}) + b(e^{\lambda_1 t} + \lambda_1 t e^{\lambda_1 t}) + cte^{\lambda_1 t} = 0 \tag{38}$$

or

$$(a\lambda_1^2 + b\lambda_1 + c)te^{\lambda_1 t} + (2a\lambda_1 + b)e^{\lambda_1} = 0 \tag{39}$$

so this is a solution when $a\lambda_1^2 + b\lambda_1 + c = 0$, that is when λ_1 satisfies the auxliary equation, and when

$$\lambda_1 = -\frac{b}{2a} \tag{40}$$

Now, if you look at the formula for solutions to the quadratic equation

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{41}$$

we see that $-b/2a$ is a solution if an only if $b^2 = 4ac$, which happens and only happens, when there is only one solution.

- **Example**: $\ddot{y} + 3\dot{y} + 2y = 0$ has auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ with roots $\lambda_1 = -1$, $\lambda_2 = -2$ so the general solution is

  $$y(t) = C_1 e^{-t} + C_2 e^{-2t} \tag{42}$$

  This corresponds to over damping.

- **Example**: $\ddot{y} + 2\dot{y} + y = 0$ has auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = -1$ and so the general solution is

  $$y(t) = (C_1 + C_2 t)e^{-t} \tag{43}$$

- **Example**: If the auxiliary equation $\lambda^2 + \lambda + 1 = 0$ with complex roots $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ the general complex solution is

  $$y(t) = C_1 e^{-\frac{1}{2}t + i\frac{1}{2}\sqrt{3}t} + C_2 e^{-\frac{1}{2}t - i\frac{1}{2}\sqrt{3}t} \tag{44}$$

  where C_1 and C_2 are complex constants. The general real solution can be obtained by imposing the constraint $C_2 = C_1^*$:

  $$y(t) = e^{-\frac{1}{2}t}\left[C_1\left(\cos\frac{1}{2}\sqrt{3}t + i\sin\cos\frac{1}{2}\sqrt{3}t\right) + C_1^*\left(\cos\frac{1}{2}\sqrt{3}t - i\sin\cos\frac{1}{2}\sqrt{3}t\right)\right] \tag{45}$$

  Writing $C_1 = \frac{1}{2}(A - iB)$ where A and B are real constants gives

  $$y(t) = e^{-\frac{1}{2}t}\left(A\cos\frac{1}{2}\sqrt{3}t + B\sin\frac{1}{2}\sqrt{3}t\right) \tag{46}$$

  this is the underdamped case, it still oscillates.