Note Fourier.2[1] 26 February 2007 as 231.II.2.tex and 16 November 2009 in this form.

Fourier Analysis

The Fourier integral

So far we have discussed the expansion of periodic functions; here we extend this discussion to functions which are not periodic. We will see that the expansion will go from a sum over a countably infinite set of basis function, to an integral over a continuum of basis function. It is possible to see why this will happen, the periodic function, with period l say, is expanded into basis functions which repeat after l, this does not require the basis functions to have period l, but, it does require that their period divides a whole number of times into l. This reduces the set of basis functions to a countably infinite set.

The Fourier integral: definition

The expansion is an integral

$$f(t) = \int_{-\infty}^{\infty} dk f\tilde{(k)} e^{ikt} \tag{1}$$

where

$$f\tilde{(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(t) e^{-ikt} \tag{2}$$

Here, the basis functions are the so called **plane waves**, $\exp{(ikt)}$ and the Fourier coefficients are now functions, $\tilde{f}(k)$. The sum over a discrete index n has been replaced by an integral of a continuous index k. No proof is given here, we will briefly revisit these formulas again after we have done the Dirac delta function. The important point is that the Fourier integral is like the Fourier series, only with a continuous basis, so the coefficient c_n with the discrete index n, is replaced by a function $\tilde{f}(k)$, that is, something with a continuous index k.

It should be noted that there are a number of different conventions for defining the Fourier integral; these vary in where the 2π goes. Some people share it between the $f(t)$ and $f\tilde{(k)}$ formula, giving factors of $1/\sqrt{2\pi}$ in front of both itegrals. Others send k to $2\pi k$ and $f\tilde{(k)}$ to $f\tilde{(k)}/2\pi$ so there are no constants multiplying either integral but the exponentials have 2πs in them.

The Fourier integral is not defined for all functions, the integrals defining $\tilde{f}(k)$ may be divergent. L^1 is often used, if

$$\int_{-\infty}^{\infty} dt |f(t)| < \infty \tag{3}$$

then the Fourier integral exists. However, the Fourier coefficient $f\tilde{(k)}$ may not itself have a Fourier integral: that is, the Fourier transform, the map from a function to its Fourier

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coefficient, is not closed on L^1. A space that works in this respect is the **Schwatz space** S, the space of rapidly decreasing functions, if $f \in S$ then $\tilde{f} \in S$. We won't go into the definition of **rapidly decreasing** here, basically, it demands that f and all its detiviative decay rapidly for large $|t|$.

- **Example:** Consider the square wave packet

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & |t| > 1 \end{cases} \tag{4}$$

This is an example of an L^1 function which is not in S. Now,

$$f(\tilde{k}) = \frac{1}{2\pi} \int_{-1}^{1} dt e^{-ikt} = \left. \frac{e^{-ikt}}{-2\pi ik} \right|_{-1}^{1} = \frac{\sin k}{\pi k} \tag{5}$$

and so

$$f(t) = \int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ikt} \tag{6}$$

Hence, for example, putting $t = 0$ gives

$$\pi = \int_{-\infty}^{\infty} dk \frac{\sin k}{k} \tag{7}$$

This is Dirichlet's integral and can also be calculated using a technique called contour integration. At $x = \pm 1$, f has a jump discontinuity, the Fourier integral averages at a jump, just like the Fourier series, we can see this in this case, at $t = 1$:

$$\int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ik} = \int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} (\cos k + i \sin k) \tag{8}$$

Now dropping the odd integral over $\sin^2 k / k$ we get

$$\int_{-\infty}^{\infty} dk \frac{\sin k}{\pi k} e^{ik} = \int_{-\infty}^{\infty} dk \frac{\sin k \cos k}{\pi k} = \int_{-\infty}^{\infty} dk \frac{\sin 2k}{2\pi k} = \frac{1}{2} \tag{9}$$

since it is in the Dirichlet integral form.