A Fourier series example[1]

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What is the Fourier series for

$$f(t) = \begin{cases} 2-t & 0<t<4 \\ t-6 & 4<t<8 \end{cases} \tag{1}$$

with $f(t+8) = f(t)$

Well, first we should note that this is an even function, you can see this by ploting it out, but we can also check by looking to see how it is defined for negative t, for example, in the interval $-4 < t < 0$ we can see from the periodicity that $f(t) = f(t+8)$ so for $t \in (-4, 0)$

$$f(t) = f(t+8) = (t+8) - 6 = t+2 \tag{2}$$

and so for $t \in (-4, 0)$ we have $f(-t) = -t+2 = f(t)$, so the function is even. You should try the other interval $(-8, -4)$ as well, but, basically, this is an even function and $b_n = 0$ for all n.

To work out the a_n;

$$a_n = \frac{1}{4}\int_0^8 f(t)\cos\frac{\pi nt}{4}dt \tag{3}$$

and, using the definition of $f(t)$ we get

$$4a_n = \int_0^4 (2-t)\cos\frac{\pi nt}{4}dt + \int_4^8 (t-6)\cos\frac{\pi nt}{4}dt \tag{4}$$

One way to finish this would be to do both integrals seperately, I will try and make it easier by doing a change of variables in the second integral in the hope that that will make it equal the first. Let $t-6 = 2-t'$, so $t = 8-t'$. Now

$$\cos\frac{\pi nt}{4} = \cos\frac{\pi n(8-t')}{4} = \cos\left(-\frac{\pi nt'}{4} + 2\pi\right) = \cos\left(-\frac{\pi nt'}{4}\right) = \cos\frac{\pi nt'}{4} \tag{5}$$

where I have used the periodicity $\cos(\theta + 2\pi) = \cos\theta$ and the evenness $\cos(-\theta) = \cos\theta$ of cosine. Now keeping track of the various signs (dt'=-dt, there is a sign from switching the order of the integral limits) it is easy to show the second integral is the same as the first and so

$$4a_n = 2\int_0^4 (2-t)\cos\frac{\pi nt}{4}dt \tag{6}$$

Lets integrate that by parts, with $u = 2-t$, so

$$2a_n = (2-t)\frac{4}{n\pi}\cos\frac{\pi nt}{4}\Big]_0^4 + \frac{4}{n\pi}\int_0^4 \sin\frac{\pi nt}{4}dt \tag{7}$$

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Now the first term has a zero from the sine at both ends, the second term needs to be integrated again to give

$$2a_n = -\frac{16}{n^2\pi^2}\cos\frac{\pi nt}{4}\Big]_0^4 \tag{8}$$

and using $cos\pi n = (-1)^n$ we get

$$a_n = \frac{16}{n^2\pi^2} \tag{9}$$

for n odd, and $a_n = 0$ for n even.