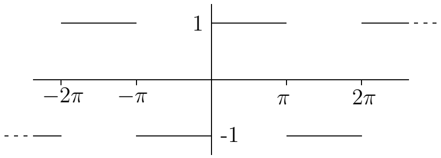
Complex Fourier series for the block wave[1] 22 October 2010

Consider the block wave with period $l = 2\pi$

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases} \tag{1}$$

with $f(t + 2\pi) = f(t)$.



Lets work out its complex Fourier series

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt \tag{2}$$

and, to substitute in for $f(t)$ we split the integration interval into the two subintervals, $[-\pi, 0]$ where $f(t) = -1$ and $[0, \pi]$ where $f(t) = 1$, hence

$$c_n = \frac{1}{2\pi}\left[-\int_{-\pi}^{0} e^{-int}dt + \int_{0}^{\pi} e^{-int}dt\right] \tag{3}$$

Now, doing the two integrals we get

$$c_n = \frac{1}{2\pi}\left\{-\frac{1}{-in}\, e^{-int}\Big]_{-\pi}^{0} + \frac{1}{-in}\, e^{-int}\Big]_{0}^{\pi}\right\} \tag{4}$$

$$= \frac{i}{2n\pi}\left[-1 + e^{-in\pi} + e^{in\pi} - 1\right] \tag{5}$$

Now $\exp(-in\pi) = \cos n\pi - i\sin n\pi = (-1)^n$; in the same way $\exp(in\pi) = (-1)^n$ so we get

$$c_n = \begin{cases} \frac{-i}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{6}$$

Now, Parseval's theorem tells us that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{7}$$

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The left hand side is just one because $|f(t)|^2 = 1$ for all t, the right hand side is

$$|c_n|^2 = c_n^* c_n = \frac{4}{n^2\pi^2} \tag{8}$$

for n odd. Hence

$$\frac{\pi^2}{4} = \sum_{n \text{ odd}} \frac{1}{n^2} = 2\sum_{n>0,\text{ odd}} \frac{1}{n^2} \tag{9}$$

where we get the last term by noting that negative n gives the same result as positive n. Hence

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \dots \tag{10}$$

Now

$$\frac{\pi^2}{8} \approx 1.23370055 \tag{11}$$

and

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} \approx 1.18386495 \tag{12}$$

add a few more terms

$$\sum_{n<=21,\text{ odd}} \frac{1}{n^2} \approx 1.21098888 \tag{13}$$

so we can see the series converging towards the right answer, but not amazingly fast.