481 Tutorial Sheet 1, Outline solutions[1]

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Questions

1. The Nernst equation was derived under the assumption that the membrane potential was negative and the ion being considered had positive charge. Rederive this result for a negatively charged ion and for the case when E is positive to verify that it applies in all these cases.

   Solution: We will begin by revising the original derivation. There are two key formula, first, if a positive ion has charge zq, the idea being that q is the charge of a proton and z is a positive integer, then the energy required to move it from potential zero to potential V is zqV. Secondly, at a temperature T the probability a particle has energy greater than $\mathcal{E}$ is

   $$\text{prob}(\text{energy} > \mathcal{E}) = e^{-\mathcal{E}/k_bT} \tag{1}$$

   This is calculated by integrating the Boltzmann probability density, as described, for example, in note 4a. Now, in the original senario the cell is at potential $V < 0$ and we are considering a positive ion with charge zq where $z > 0$. Now the potential gap does not prevent ions diffusing in to the cell, but it does prevent them from diffusing out; only ions with enough energy to cross the gap can diffuse out. Since V is negative the potential gap is $-zqV > 0$ and the proportion of cells with energy greater than $-zqV$ is $\exp zqV/k_bT = \exp zV/V_T$ where we have used the notation $V_T = k_bT/q$. Since the diffusive flow is proportion to the concentration of ions energetically capable of crossing the gap, the flow in is proportional to ρ_o, the density outside and flow out is proportional to $\rho_i \exp zV/V_T$. Now, the equilibrium potential $V = E$ is the potential for which the two flows balance and hence

   $$\rho_o = \rho_i e^{zE/V_T} \tag{2}$$

   or

   $$E = \frac{V_T}{z} \log \frac{\rho_o}{\rho_i} \tag{3}$$

   which is the Nernst equation.

   Now, to actually address the question, say instead the ion was negatively charged, $z < 0$, so that there is no potential barrier to it flowing out of the cell, but there is one to it flowing in. The energy barrier is $zqV > 0$ and hence the flow in is proportional to $\rho_o \exp(-zV/V_T)$ and at equilibrium potential

   $$\rho_i = \rho_o e^{-zE/V_T} \tag{4}$$

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or, as before,

$$E = \frac{V_T}{z} \log \frac{\rho_o}{\rho_i} \tag{5}$$

If V is positive and the ion is positive then the equation is still

$$\rho_i = \rho_o e^{-zE/V_T} \tag{6}$$

since the positive ion has a potential barrier of zqV to entering the cell and, if the potential is positive and the ion negative, the equation is

$$\rho_o = \rho_i e^{zE/V_T} \tag{7}$$

Hence, the Nernst equation is the same in each case, even though z and V have different signs.

2. Consider the effect of a triangular pulse on the integrate and fire neuron. When does this cause a spike?

$$I_e = \begin{cases} At & t \in (0, T) \\ A(2T - t) & t \in (T, 2T) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Solution: This question is just a big long annoyance, the challenge is to solve the integrate and fire equation

$$\tau_m \dot{V} = E_L - V + R_m I_e \tag{9}$$

with a different input current. To save on writing lets absorb R_m into A and consider the situation when $t < T$:

$$\tau_m \dot{V} = E_L - V + At \tag{10}$$

and hence

$$\dot{V} + \frac{1}{\tau_m} V = \frac{1}{\tau_m} (E_L + At) \tag{11}$$

or

$$\frac{d}{dt} \left(V e^{t/\tau_m} \right) = \frac{1}{\tau_m} (E_L + At) e^{t/\tau_m} \tag{12}$$

and so, integrating by parts we get

$$V = E_L + A(t - \tau) + Ce^{-t/\tau_m} \tag{13}$$

so, assuming the voltage starts at E_L. the resting value, we have

$$E_L = E_L - A\tau + C \tag{14}$$

and hence $C = A\tau$ giving

$$V = E_L + At + A\tau \left(e^{-t/\tau_m} - 1 \right) \tag{15}$$

If $V(T) > V_t$, the threshold value, then it spikes before $t = T$; hence we have a spike if

$$V_t < E_L + AT + A\tau e^{-T/\tau_m} - A\tau \tag{16}$$

or

$$A > \frac{V_t - E_L}{\exp\left(-T/\tau_m\right) + T - \tau} \tag{17}$$

On the otherhand, even if it didn't spike as the current increased, it could spike as the current falls. For $T < t < 2T$

$$\dot{V} + \frac{1}{\tau_m}V = \frac{1}{\tau_m}\left(E_L + 2AT - At\right) \tag{18}$$

so, by analogy with the previous solution, we have

$$V = E_L + 2AT - A(t - \tau) + Ce^{-t/\tau_m} \tag{19}$$

and setting, $t = T$ and, using the value of $V(T)$ calculated above, we have

$$C = A\tau\left(1 - 2e^{T/tau}\right) \tag{20}$$

Substituting this back in we have

$$\tau\dot{V} = -A\tau - A\tau e^{-t/\tau} + 2A\tau e^{(T-t)/tau} \tag{21}$$

so $\dot{V} = 0$ if $t = t_m$ where

$$t_m = \tau\log\left(2e^{T/\tau} - 1\right) \tag{22}$$

and there is a spike if $V(t_m) > V_t$ if $t_m < 2T$ or if $V(2T) > V_t$ otherwise. As an exam question, this would be a good enough question if I had left out the down slope, that is, if the current had been

$$I_e = \begin{cases} At & t \in (0, T) \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

but this way it is too tedious to ask in an exam.

3. Another model of the synaptic conductance has an auxiliary function z and satisfies

$$\begin{aligned} \tau_s\dot{P}_s &= eP_m z - P_s \\ \tau_z\dot{z} &= -z \end{aligned} \tag{24}$$

with the rule that z is set to one whenever a spike arrives. P_m is a constant. Solve this for the response to single spike, both with $\tau_s = \tau_z$ and otherwise. In the $\tau_s = \tau_z$ case consider the maximum value of P_s and how this changes if two spikes arrive one after the other. Speculate on the physiological meaning of z.

Solution: This is a much nicer question: first, the biology, it is easy to surmise that z is the level of neurotransmitter in the cleft so it goes instantaneously to one when the spike arrives. P_s relaxes towards the value of z but z is itself falling because of the re-uptake pumps reabsorb the neurotransmitter. This differs from the model we looked at in the lectures where the neurotransmitter had a block wave form, it rose instantaneously to its maximum value, sustained that value and then disappeared.

Now, z is set to one whenever a spike arrives, so if a spike arrives at time $t = 0$

$$z = e^{-t/\tau_z} \tag{25}$$

Substituting back in to the equation for P_s we have

$$\dot{P}_s + frac1\tau_s P_s = \frac{e}{\tau_s} P_m e^{-t/\tau_z} \tag{26}$$

Hence

$$\frac{d}{dt}\left(P_s e^{t/\tau_s}\right) = \frac{e}{\tau_s} P_m e^{t/\tau_s - t/\tau_z} \tag{27}$$

and, if $\tau_s \neq \tau_z$ we integrate to get

$$P_s = -\frac{\tau_z}{\tau_s - \tau_z} P_m e^{-t\tau_z} + Ce^{-t\tau_s} \tag{28}$$

where I have used

$$\left(\frac{1}{\tau_s} - \frac{1}{\tau_z}\right)^{-1} = -\frac{\tau_s \tau_z}{\tau_s - \tau_z} \tag{29}$$

and we have switched the terms in the denominator because τ_s is typically bigger than τ_z. Finally, if this is the first spike in a long time, $P_s(0) = 0$ so

$$C = \frac{\tau_z}{\tau_s - \tau_z} P_m \tag{30}$$

and

$$P_s = \frac{\tau_z}{\tau_s - \tau_z} P_m \left(e^{-t\tau_s} - e^{-t\tau_z}\right). \tag{31}$$

If $\tau_s = \tau_z$ then the integration is slightly different, let $\tau = \tau_s = \tau - z$ so

$$\frac{d}{dt}\left(P_s e^{t/\tau}\right) = \frac{e}{\tau} P_m \tag{32}$$

and

$$P_s = \frac{e}{\tau} P_m t e^{-t/\tau} + Ce^{-t/\tau} \tag{33}$$

and, setting $P_s(0) = 0$ gives $C = 0$ so

$$P_s = \frac{1}{\tau} P_m t e^{1-t/\tau} \tag{34}$$

Substituting this back in to the equation for $\dot{P}_s$ gives

$$\tau \dot{P}_s = P_m \left(1 - \frac{t}{\tau}\right) e^{1-t/\tau} \tag{35}$$

so the maximum occurs at $t = \tau$ and, calculating $P_s(\tau)$ shows that the maximum value is P_m.

If $P_s(0) = P_0 > 0$ as would happen if one spike followed another, we would have $C = P_0$ and hence

$$P_s = \frac{1}{\tau} P_m t e^{1-t/\tau} + P_0 e^{-t/\tau} \tag{36}$$

and

$$\tau \dot{P}_s = P_m \left(1 - \frac{t}{\tau} - \frac{P_0}{eP_m} \right) e^{1-t/\tau} \tag{37}$$

In this case, the maximum occurs when

$$1 - \frac{t}{\tau} - \frac{P_0}{P_m e} = 0 \tag{38}$$

or $t = (1 - f)\tau$ where $f = P_0/P_m e$. Substituting back into the equation for P_s we get the amazing result

$$P_s = P_m e^f \tag{39}$$

so if $P_0 = 0$ the maximum is P_m as before, if $P_0 = P_m$ then the maximum $P_m e^{1/e} \approx 1.44 P_m$.