231 Outline Solutions Tutorial Sheet 7, 8 and 9.[1][2]

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Problem Sheet 7

1. Which of the following vector fields are conservative?

   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.

   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ where $\mathbf{B}$ is a constant vector.

   Solution:

   (a) $\mathbf{F} = \nabla yz\cos x$ so $\mathbf{F}$ is conservative.

   (b) curl $\mathbf{F} = \mathbf{k} \neq 0$ so $\mathbf{F}$ is not conservative.

   (c) A short calculation gives curl $\mathbf{F} = \mathbf{B}$ so $\mathbf{F}$ is not conservative. Remark: $\frac{1}{2}(\mathbf{B}\times\mathbf{r})$ is a vector potential for the constant vector field $\mathbf{B}$.

2. Find a potential for

$$\mathbf{F} = (e^{yz} + zye^{xy})\mathbf{i} + (xze^{yz} + xze^{xy})\mathbf{j} + (xye^{yz} + e^{xy})\mathbf{k} \tag{1}$$

   *Solution:*So, as before

$$\mathbf{F} = \nabla\phi \tag{2}$$

   implies $F_1 = \partial_x\phi$ and hence

$$\partial_x\phi = e^{yz} + zye^{xy} \tag{3}$$

   so integrating

$$\phi = xe^{yz} + ze^{xy} + C(y,z) \tag{4}$$

   where $C(y,z)$ is a function of y and z to be determined, in fact, it turns out to be constant, it is easy to check

$$\phi = xe^{yz} + ze^{xy} + C \tag{5}$$

   with C a constant, is a potential for the field.

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[2]Including material from Chris Ford, to whom many thanks.

3. Compute the flux of the vector field $\mathbf{F} = (x + x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \le z \le 1$.

   *Solution:*As we have seen before, $\mathbf{r}(u, v) = \cos u\mathbf{i} + \sin u\mathbf{j} + v\mathbf{k}$ giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos u\mathbf{i} + \sin u\mathbf{j}. \tag{6}$$

   and

$$\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (x + x^2)\cos u + y \sin u = \cos^2 + \cos^3 u + \sin^2 u = 1 + \cos^3 u. \tag{7}$$

   and so, noting that the $\mathbf{F}$ is perpendicular to the normal at both ends and so we need only include the curved surface

$$\int_S \mathbf{F} \cdot \mathbf{dA} = \int_0^1 dv \int_0^{2\pi} du \ (1 + \cos^3 u) = 2\pi, \tag{8}$$

   since the $\cos^3 u$ integral is zero by symmetry.

   Note: This problem can be solved by noting that $x^2\mathbf{i}$ makes no contribution (by symmetry) and $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has flux 2π since $\mathbf{F} \cdot \mathbf{n} = 1$.

4. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the $z = 0$ plane.

   *Solution:*To parametrize the sphere choose $(u, v) = (\theta, \phi)$, that is, spherical polar angles. Since the radius $r = a$ this gives $x(u, v) = a \sin u \cos v$, $y(u, v) = a \sin u \sin v$, $z(u, v) = a \cos u$ with $0 \le \theta \le \pi/2$ and $0 \le \phi < 2\pi$. Since $\mathbf{F}$ only has a non-zero z-component we just need

$$\begin{aligned}
\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)_3 &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\
&= a^2(\cos u \cos v \sin u \cos v + \cos u \sin v \sin u \sin v) \\
&= a^2 \cos u \sin u,
\end{aligned} \tag{9}$$

   which is positive. So the orientation is upwards. Now $F_z = z^3 = a^3 \cos^3 u$, and so

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{A} &= a^5 \int_0^{\frac{1}{2}\pi} du \int_0^{2\pi} dv \ \cos^4 u \sin u \\
&= 2\pi a^5 \int_0^{\frac{1}{2}\pi} du \ \cos^4 u \sin u \\
&= 2\pi a^5 \cdot -\frac{\cos^5 u}{5}\bigg]_0^{\frac{1}{2}\pi} = \frac{2\pi a^5}{5}.
\end{aligned} \tag{10}$$

Problem Sheet 8

1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (1-u^2)\mathbf{k} \tag{11}$$

with $1 \le u \le 2$ and $0 \le v \le 2\pi$, oriented to give a positive answer.

*Solution:*So,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \cos v\mathbf{i} + \sin v\mathbf{j} - 2u\mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -u\sin v\mathbf{i} + u\cos v\mathbf{j} \end{aligned} \tag{12}$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 2u^2\cos v \\ 2u^2\sin v \\ u \end{pmatrix} \tag{13}$$

Now, on the parabola

$$\mathbf{F} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + \mathbf{k} \tag{14}$$

so, taking the dot product, the flux, ϕ, is

$$\begin{aligned} \phi &= \int_1^2 du \int_0^{2\pi} dv \left(2u^3\cos^2 v + 2u^3\sin^2 v + u^2\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned} \tag{15}$$

which is a positive answer, so the orientation was correct.

2. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{16}$$

with $0 \le u \le 5$ and $0 \le v \le 2\pi$, oriented to give a positive answer.

*Solution:*And again with the paraboloid:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -2\sin v\mathbf{i} + \cos v\mathbf{j} \end{aligned} \tag{17}$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -\cos v \\ -2\sin v \\ 0 \end{pmatrix} \tag{18}$$

On the parabola

$$\mathbf{F} = e^{-\sin v}\mathbf{i} - \sin v\mathbf{j} + 2\cos v \sin u\mathbf{k} \tag{19}$$

and the flux is

$$\phi = \int_0^5 du \int_0^{2\pi} dv \left(-\cos v e^{-sinv} + 2\sin^2 v\right) \tag{20}$$

Now,

$$\int_0^{2\pi} dv \cos v e^{-sinv} = 0 \tag{21}$$

there are lots of ways to see this, one way is to note that the integrand is odd about the point $v = \pi/2$ and a change of variable and the periodicity could be used to make the integral symmetric about this point

$$\int_0^{2\pi} dv \cos v e^{-sinv} = \int_{-\pi/2}^{3\pi/2} dv \cos v e^{-sinv} \tag{22}$$

and then let $w = v - \pi/2$. This leaves the other bit of the integral, which we do using the usual

$$2\sin^2 x = 1 - \cos 2x \tag{23}$$

giving

$$\phi = 10\phi \tag{24}$$

3. Use Green's Theorem to evaluate

$$\oint_c (y^2 dx + x^2 dy) \tag{25}$$

where C is the square with vertice $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and oriented anti-clockwise.

*Solution:*By Green's theorem

$$\oint_c (y^2 dx + x^2 dy) = \int_0^1 dx \int_0^1 dy(2x - 2y) = \int_0^1 dx(2x - 1) = 0 \tag{26}$$

4. Calculate directly and using Stoke's Theorem

$$\int_S \mathbf{F} \cdot \mathbf{dS} \tag{27}$$

where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.

*Solution:*So, to calculate directly, choose some parameterization

$$\mathbf{r} = \rho \cos\phi\mathbf{i} + \rho \sin\phi\mathbf{j} + (9 - \rho^2)\mathbf{k} \tag{28}$$

works. Now

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \phi} &= -\rho \sin\phi \mathbf{i} + \rho \cos\phi \mathbf{j} \\
\frac{\partial \mathbf{r}}{\partial \phi} &= \cos\phi \mathbf{i} + \sin\phi \mathbf{j} - 2\rho \mathbf{k}
\end{aligned} \tag{29}
$$

and, choosing the other order to make the normal upward points

$$
\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 2\rho^2 \cos\phi \\ 2\rho^2 \sin\phi \\ -\rho \end{pmatrix} \tag{30}
$$

Now, writing this as $(2\rho x, 2\rho y, \rho)$ and doing the dot product with $\mathbf{F}$ we are left with only terms which are linear in x or y and since the ϕ integral goes all the way around, we see the answer is zero.

Next, using Stokes

$$
\int_S \operatorname{curl} \mathbf{A} \cdot \mathbf{dS} = \oint_c \mathbf{A} \cdot \mathbf{dl} \tag{31}
$$

hence, to apply Stokes, we have to write $\mathbf{F}$ as $\operatorname{curl} \mathbf{A}$, in other words, find a vector potential for $\mathbf{F}$. It is easy to check that $\operatorname{div} F = 0$ so this should be possible. We will use the formula that was used to prove the existence of vector potential for divergenceless fields on star-shapped domains. Hence

$$
\begin{aligned}
\mathbf{A} &= \int_0^1 \mathbf{F}(t\mathbf{r}) \times t\mathbf{r} \\
&= \frac{1}{3} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z-y & z+x & -(x+y) \\ x & y & z \end{vmatrix} \\
&= \frac{1}{3} \begin{pmatrix} z^2 + xz + xy + y^2 \\ -x^2 - xy - z^2 + yz \\ zy - y^2 - xz - x^2 \end{pmatrix}
\end{aligned} \tag{32}
$$

where I got the third by noting that the overall factor of t^2 came out of the determinant, and then integrating it. Since this formula is complicated it would certainly be a good idea to check $\mathbf{F} = \nabla \times \mathbf{F}$.

Now, to apply Stoke's theorem:

$$
\int_S \operatorname{curl} \mathbf{A} \cdot \mathbf{dS} = \oint_c \mathbf{A} \cdot \mathbf{dl} \tag{33}
$$

where C is the circle of radius three around the origin in the xy-plane: $x^2 + y^2 = 9$ and $z = 0$. We parameterize with

$$
\mathbf{r} = \cos t \mathbf{i} + \sin t \mathbf{j} \tag{34}
$$

so that

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} \tag{35}$$

Restricting **A** to the curve and doing the dot product gives

$$\oint_c \mathbf{A} \cdot \mathbf{dl} = \int_0^{2\pi} (-cs^2 - s^3 - c^3 - c^2 s)dt = 0 \tag{36}$$

where $c = \cos t$ and $s = \sin t$ and we are using the usual anti-symmetry arguement that odd powers of sine and cosine integrate to zero over their entire period.

Problem Sheet 9

1. Let S be the closed surface consisting of the portion of the paraboloid $z = x^2 + y^2$ for which $0 \leq z \leq 1$ and capped by the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j} - y\mathbf{k}$ in the outrward direction across S.

   *Solution:*Use Gauss's theorem and the fact that $\operatorname{div} \mathbf{F} = 0$ to get zero.

2. For any conduction loop C the electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$ are related by

$$\oint_c \mathbf{E} \cdot d\mathbf{r} = -\int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{37}$$

   where $C = \delta S$, use Stoke's theorem to find a differential equation relating $\mathbf{E}$ and $\mathbf{B}$.

   *Solution:*By Stoke's theorem

$$\int\int_S \nabla \times \mathbf{E} \cdot d\mathbf{r} = -\int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{38}$$

   or

$$\int\int_S \left(\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} \tag{39}$$

   and since this is true for any simple S it must hold, by continuity, that

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{40}$$

3. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

   *Solution:*Let S be the closed surface comprising the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the disk (needed to close the surface) $z = 0$, $x^2 + y^2 \leq 1$. Using Gauss' theorem the flux of $\mathbf{F}$ out of S is

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_D \operatorname{div} \mathbf{F} \; dV = 3 \int_D (x^2 + y^2 + z^2) \; dV,$$

   where D is the region enclosed by S. This integral can be worked out through spherical polar coordinates or just by splitting D into small spherical half-shells of volume $2\pi r^2 \delta r$:

$$\int_S \mathbf{F} \cdot d\mathbf{A} = 2\pi \int_0^1 dr \; r^2 \; 3r^2 = \frac{6\pi}{5}.$$

   Now the flux out of the disk is zero since here $\mathbf{F}$ is perpendicular to the outward normal $\mathbf{n} = -\mathbf{k}$. Thus the flux through the hemisphere is $6\pi/5$. The flux out of the whole sphere is $12\pi/5$.

4. Consider, again, the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

(a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.

(b) Compute the flux of $\mathbf{F}$ out of the box $1 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 1$.

(c) Compute the flux of $\mathbf{F}$ out of the box $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$.

Solution:

(a) Flux integral trivial since $\mathbf{F} \cdot \mathbf{n}$ is constant over the sphere ($\mathbf{n}$ is the outward normal). Here $\mathbf{F} \cdot \mathbf{n} = 1/a^2$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{A} = 4\pi a^2 \, \frac{1}{a^2} = 4\pi$. Note that this is independent of the radius of the sphere.

(b) We know div$\mathbf{F} = 0$. Therefore Gauss' theorem implies that the flux out of the box is zero. Note that this result is very difficult to obtain via a direct surface integral computation.

(c) Naively, Gauss' theorem also gives a zero flux in this case. However, this is not correct. Indeed, an indiscriminate use of Gauss' theorem would also give a zero flux in part i). The point is that in parts i) and iii) (but not ii)) the 'inside' region contains the origin where $\mathbf{F}$ is singular. As in part i) the correct answer to this question is 4π. This can be obtained by combining Gauss' theorem with the result of the direct calculation of part a). To do this remove a sphere (centred at the origin) of radius less than one from the box. In this region $\mathbf{F}$ is smooth and so Gauss' theorem implies that the flux out of this region is zero. However this flux comprises two parts; the flux out of the box (which we wish to compute) and the flux into the cut sphere (which from part i) is -4π). Therefore the flux out of the box must be 4π.

5. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

*Solution:*Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 dt \; \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$. Now $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k} = \mathbf{r} - 3z\mathbf{k}$ so that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 dt \; -3zt\mathbf{k} \times \mathbf{r}t = -3\int_0^1 dt \; t^2 \; (zx\mathbf{j} - zy\mathbf{i}) = zy\mathbf{i} - zx\mathbf{j}.$$