231 Outline Solutions Tutorial Sheet 4, 5 and 6.[1,2]

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Problem Sheet 4

1. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is
$$J = r^2\sin\theta.$$
Consider the hemisphere defined by
$$\sqrt{x^2+y^2+z^2} \le 1, \qquad z \ge 0.$$
Using spherical polar coordinates compute its volume and centroid.

   *Solution:*Spherical polar coordinates are defined by
$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{1}$$
The Jacobian is
$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}. \\ &= r^2\sin^2\theta\left[\cos^2\theta\cos^2\phi + \cos^2\theta^2\sin^2\phi^2 + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right] \\ &= r^2\sin\theta. \end{aligned} \tag{2}$$
Volume $= \int_D\, dV$. Centroid $\bar{x} = \bar{y} = 0$ by symmetry and $\bar{z} = \int_D\, dV\ z / \int_D\, dV$. Now
$$\begin{aligned} \int_D dV &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_0^1 dr\ r^2\sin\theta = 2\pi\int_0^{\pi/2} d\theta\ \sin\theta\ \frac{1}{3}. \\ &= -\frac{2}{3}\pi\cos\theta\Big|_0^{\pi/2} = 2\pi/3 \end{aligned} \tag{3}$$
as expected.

   The other integral is
$$\begin{aligned} \int_D dV\ z &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} d\theta \int_0^1 dr\ r^2\sin\theta\cdot r\cos\theta = 2\pi\int_0^{\frac{1}{2}\pi} d\theta\ \sin\theta\cos\theta\ \frac{1}{4} \\ &= \frac{\pi}{2}\int_0^{\frac{1}{2}\pi} d\theta\ \frac{1}{2}\sin 2\theta = \frac{\pi}{4} \end{aligned} \tag{4}$$
and therefore $\bar{z} = 3/8$.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

2. Show $\operatorname{div}\mathbf{r} = 3$ and $\operatorname{grad}|\mathbf{r}| = \mathbf{r}/|\mathbf{r}|$.

   *Solution:*Well

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{5}$$

   and so

$$\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3 \tag{6}$$

   and

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k} \tag{7}$$

   and

$$\frac{\partial}{\partial x}r = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \tag{8}$$

   by the chain rule.

3. Find $\nabla(1/|\mathbf{r}|)$.

   *Solution:*So this is similar to the previous one

$$\nabla \frac{1}{r} = \frac{\partial}{\partial x}\frac{1}{r}\mathbf{i} + \frac{\partial}{\partial y}\frac{1}{r}\mathbf{j} + \frac{\partial}{\partial z}\frac{1}{r}\mathbf{k} \tag{9}$$

   and

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial}{\partial x}r = -\frac{x}{r^3} \tag{10}$$

   hence

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3} \tag{11}$$

   and this, or course, is consistient with $\operatorname{curl}\mathbf{r}/r^3 = 0$.

4. Show $\operatorname{grad} f(r) = f'(r)\hat{\mathbf{r}}$ where $r = |\mathbf{r}|$. If $\mathbf{F}(r) = f(r)\mathbf{r}$ find $\operatorname{div}\mathbf{F}(r)$. Find $\operatorname{div}\operatorname{grad} f(r)$.

   *Solution:*So, now, we use the chain rule to show

$$\frac{\partial}{\partial x}f(r) = f'(r)\frac{\partial r}{\partial x} = \frac{xf'(r)}{r} \tag{12}$$

   and, since the gradient has three terms of this form, it is easy to see $\operatorname{grad} f(r) = f'(r)\hat{\mathbf{r}}$. As for the divergence, $F_1 = xf(r)$ and

$$\frac{\partial}{\partial x}xf(r) = f(r) + \frac{x^2 f'}{r} \tag{13}$$

   and so, adding three similar terms together, we get

$$\nabla \cdot \mathbf{F} = 3f + rf' \tag{14}$$

Finally, , we know the grad $f(r)$ and, so, using f'/r for f in the divergence formula we get

$$\triangle f(r) = \frac{3f'}{r} + r\left(\frac{f'}{r}\right)' = \frac{2f'}{r} + f'' \tag{15}$$

which gives us a formula for the laplacian of a spherically symmetric field in polar coördinates, later we will see how to convert partial differential equations from one coördinate system to another.

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{16}$$

is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$).

*Solution:*Note that $\mathbf{F} = \text{grad}\,(-1/r)$ and so curl $\mathbf{F} = 0$. This can also be done by direct calculation.

Problem Sheet 5

1. Calculate $\operatorname{curl} \mathbf{r}/r$ and $\operatorname{div} \mathbf{r}/r$ away from the origin. What is $\triangle r$?

   *Solution:*So it is easy enough to check these by hand, for example, the bfi component of $\operatorname{curl} \mathbf{r}/r$ is given by

$$\left(\nabla \times \frac{\mathbf{r}}{r}\right)_1 = \frac{\partial}{\partial y}\frac{z}{r} - \frac{\partial}{\partial z}\frac{y}{r} = 0 \tag{17}$$

   and, with the other two components similar $\operatorname{curl} \mathbf{r}/r = 0$. As for the divergence

$$\frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} \tag{18}$$

   and hence $\operatorname{div} \mathbf{r}/r = 2/r$. Finally, $\operatorname{grad} r = bfr/r$ so $\triangle r = 2/r$.

2. Prove the identity

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{19}$$

   *Solution:*So this is easy by direct calculation,

$$\nabla \cdot (\nabla \times \mathbf{F}) = \partial_x(\partial_y F_3 - \partial_z F_2) + \partial_y(\partial_z F_1 - \partial_x F_3) + \partial_z(\partial_x F_2 - \partial_y F_1) \tag{20}$$

   and expanding out, all the terms cancel, assuming the partial derivative commute.

3. Prove the identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \triangle \mathbf{F}. \tag{21}$$

   *Solution:*Lets do the first component:

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = \frac{\partial}{\partial y}(F_{2,x} - F_{1,y}) - \frac{\partial}{\partial z}(F_{3,x} - F_{1,z}) \tag{22}$$

   where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and I am using a comma notation for differenciation so for example

$$F_{2,x} = \frac{\partial F_2}{\partial x} \tag{23}$$

   Now, taking away some brackets

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = F_{2,xy} - F_{1,yy} - F_{3,xz} - F_{1,zz} \tag{24}$$

   Coming from the other side

$$[\nabla(\nabla \cdot \mathbf{F})]_1 = \frac{\partial}{\partial x}(F_{1,x} + F_{2,y} + F_{3,z}) = F_{1,xx} + F_{2,yx} + F_{3,zx} \tag{25}$$

   so

$$[\nabla \times (\nabla \times \mathbf{F})]_1 - [\nabla(\nabla \cdot \mathbf{F})]_1 = F_{1,xx} + F_{1,yy} + F_{1,zz} = [\triangle \mathbf{F}]_1 \tag{26}$$

   and similarily for the other components.

4. Compute the line integrals:

   (a) $\int_C \ (dx\ xy \ + \frac{1}{2}dy\ x^2 \ + dz)$ where C is the line segment joining the origin and the point $(1,1,2)$.

   (b) $\int_C \ (dx\ yz \ + dy\ xz \ + dz\ yx^2)$ where C is the same line as in the previous part.

   *Solution:*A quick way here is to note that $\mathbf{F}$ is conservative.

$$\mathbf{F} = xy\mathbf{i} + \frac{1}{2}x^2\mathbf{j} + \mathbf{k} = \nabla\phi \tag{27}$$

   where $\phi = \frac{1}{2}x^2y + z$. Hence

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \phi(1,1,2) - \phi(0,0,0) = \frac{5}{2}. \tag{28}$$

   For the next part, use the parametrization $x(u) = u$, $y(u) = u$, $z(u) = 2u$ $(0 \le u \le 1)$.

$$\begin{aligned}\frac{d\mathbf{r}}{du} &= \mathbf{i}+\mathbf{j}+2\mathbf{k},\\ \mathbf{F}\cdot\frac{d\mathbf{r}}{du} &= 2u^2 + 2u^2 + 2u^3 = 4u^2 + 2u^3\end{aligned} \tag{29}$$

   so

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 du\ \left(4u^2 + 2u^3\right) = \frac{4}{3} + \frac{1}{2} = \frac{11}{6}. \tag{30}$$

5. For each of the following vector fields compute the line integral $\oint_C \mathbf{F}\cdot\mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2y\mathbf{j}$.

   *Solution:*In the first part $\mathbf{F} = \nabla\frac{1}{2}(x^2+y^2)$ so that $\mathbf{F}$ is conservative giving $\oint_C \mathbf{F}\cdot\mathbf{dl}$. In the second part parametrize curve:

$$\begin{aligned} x(u) &= \cos u\\ y(u) &= \sin u\\ z(u) &= 0\end{aligned} \tag{31}$$

   where $0 \le u \le 2\pi$ or $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}$. Now

$$\frac{d\mathbf{r}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j}. \tag{32}$$

   and

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{du} = -y\sin u - x^2y\cos u = -\sin^2 u - \cos^3 u\sin u. \tag{33}$$

   Thus

$$\oint_C \mathbf{F}\cdot\mathbf{dl} = \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^3 u\sin u\right) = -\pi, \tag{34}$$

   since the average value of $\sin^2 u$ is $\frac{1}{2}$ and $\int_0^{2\pi}\ du\ \cos^3 u\sin u = 0$ by symmetry.

Problem Sheet 6

1. For each of the following vector fields compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{l}$ where C is the semi-circle of radius two around the origin in the xy-plane with y positive, taken anti-clockwise.

   (a) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$

   (b) $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

   Solution: So, once again we parameterize the curve:

$$\mathbf{r} = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} \tag{35}$$

so

$$\frac{d\mathbf{r}}{dt} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} \tag{36}$$

and, so, for the first part, on the curve, $x = 2\cos t$ and $y = 2\sin t$ so

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -8\cos^2 t\sin t + 8\cos^2 t\sin t = 0 \tag{37}$$

and hence the integral is zero. For the second part

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 4\cos^2 t + 4\sin^2 t = 4 \tag{38}$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^\pi \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = 4\int_0^\pi dt = 4\pi \tag{39}$$

2. Evaluate the line integrals $\int_C \mathbf{F}\cdot d\mathbf{l}$ for

   (a) $\mathbf{F} = (x^2y, 4, 0)$ with C given by $\mathbf{r}(t) = (\exp(t), \exp(-t), 0)$ with t going from zero to one;

   (b) $\mathbf{F} = (z, x, y)$ with C given by $\mathbf{r}(t) = (\sin t, 3\sin t, \sin^2 t)$ with $t$going from zero to $\pi/2$.

   (c) $\mathbf{F} = \lambda(x, y)$ with $\lambda = (x^2+y^2)^{-3/2}$ and with C given by $\mathbf{r}(t) = (e^t\sin t, e^t\cos t)$ with t going from zero to one.

   Solution: For the first one

$$\mathbf{r} = (\exp(t), \exp(-t), 0) \tag{40}$$

so

$$\frac{d\mathbf{r}}{dt} = (\exp(t), -\exp(-t), 0) \tag{41}$$

and, on the curve,

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = e^{2t} - 4e^{-t} \tag{42}$$

and hence

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_0^1 (e^{2t} - 4e^{-t})dt = \frac{1}{2}e^2 + 4e^{-1} - \frac{9}{2} \tag{43}$$

For the next one

$$\mathbf{r} = (\sin t, 3\sin t, \sin^2 t) \tag{44}$$

so

$$\frac{d\mathbf{r}}{dt} = (\cos t, 3\cos t, 2\sin t\cos t) \tag{45}$$

and, on the curve,

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = (7\sin^2 t + 3\sin t)\cos t \tag{46}$$

and hence

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= \int_0^{\pi/2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \\ &= \int_0^{\pi/2} (7\sin^2 t + 3\sin t)\cos t dt = \int_0^1 (7u^2 + 3u)du = \frac{23}{6}\end{aligned} \tag{47}$$

where we have used a substitution $u = \sin t$.

Finally

$$\mathbf{r} = \big(e^t\sin t, e^t\cos t\big) \tag{48}$$

which is an exponential spiral with

$$\frac{d\mathbf{r}}{dt} = \big(e^t(\sin t + \cos t), e^t(\cos t - \sin t)\big) \tag{49}$$

and, thanksfully, this is simplifies on the curve to

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = e^{-t} \tag{50}$$

and hence

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_0^1 e^{-t}dt = e^{-1} - 1 \tag{51}$$

3. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla\phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = 3y^2\mathbf{i} + 6xy\mathbf{j}$

   (c) $\mathbf{F} = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$

   (d) $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j}$

*Solution:*So, in the first case, it is easy to see the curl is zero, having done that we want $\mathbf{F} = \nabla\phi$, hence $F_1 = \phi_{,x}$ or

$$\frac{\partial}{\partial x}\phi = x \tag{52}$$

and hence $\phi = x^2/2 + C(y,z)$, where $C(y,z)$ is an arbitrary function of y and z, substitute that back in to get

$$\frac{\partial}{\partial y}C = y \tag{53}$$

giving $\phi = x^2/2 + y^2/2 + C(z)$ where $C(z) = C$ a constant follows from $F_3 = 0$.

For the next one the curl is again zero so there is a potential,

$$\partial_x\phi = 3y^2 \tag{54}$$

so $\phi = 3y^2x + C(y,z)$. Substituting into the y equation gives

$$\partial_y\phi = 6xy + \partial_y C(y,z) = 6xy \tag{55}$$

and hence $\partial_y C(y,z) = 0$ so $C(y,z) = C(z)$, further substituting this into $\partial_z\phi = 0$ shows $C(z) = C$ a constant and $\phi = 3y^2x + C$.

For the next one the curl is again zero so there is a potential,

$$\partial_x\phi = e^x\cos y \tag{56}$$

so $\phi = e^x\cos y + C(y,z)$. Substituting into the y equation and z equation show that $C(y,z) = C$ a constant and $\phi = e^x\cos y + C$.

Finally the last one also has zero curl and

$$\partial_x\phi = \cos y + y\cos x \tag{57}$$

giving $\phi = x\cos y + y\sin x + C(y,z)$ and, again, substituting in to the y equation and z equation show that $C(y,z) = C$ a constant and $\phi = x\cos y + y\sin x + C$. It won't always work out like this with arbitrary function turning out to be an arbitrary constant, it is just an accident that I ask you three examples like this!

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}.$$

Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.

Solution:

$$\nabla \times \mathbf{F} \;=\; \frac{1}{2}\mathbf{k}\left[\partial_x\left(\frac{-x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\mathbf{k}\left[-\frac{1}{x^2+y^2}+\frac{2x^2}{(x^2+y^2)^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}\right] \\
&= 0. \qquad (58)
\end{aligned}
$$

To show that $\mathbf{F}$ is not conservative consider $\oint_C \mathbf{F}\cdot\mathbf{dl}$ where C is the unit circle. Using the obvious parametrization

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{dl} &= \int_0^{2\pi} du\ \left(-\sin^2 u-\cos^2 u\right) \\
&= -2\pi \neq 0, \qquad (59)
\end{aligned}
$$

therefore $\mathbf{F}$ is not conservative.

The domain $x^2+y^2 \geq \frac{1}{2}$, $y \geq 0$ is simply connected and $\mathbf{F}$ is irrotational and smooth is the domain. Thus $\mathbf{F}$ is conservative.

Write $\mathbf{F} = \nabla\phi$. Seek a $\phi(x,y)$ such that

$$
\frac{\partial\phi}{\partial x} = \frac{y}{x^2+y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}. \qquad (60)
$$

Integrate first equation by treating y as a constant

$$
\phi(x,y) = y\int\frac{dx}{x^2+y^2} = \tan^{-1}\frac{x}{y} + C(y). \qquad (61)
$$

Assume that x and y are non-negative, then

$$
\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = \frac{\pi}{2},
$$

so that $\phi(x,y) = -\tan^{-1}\frac{y}{x}+$ a possibly y-dependent constant. However it is easy to check that $\phi = -\tan^{-1}\frac{y}{x}$ satisfies $\frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}$. Clearly, $\tan^{-1}\frac{y}{x}$ is the usual polar angle θ, that is $\phi = -\theta$.

Can try to extend this back to the original domain $x^2+y^2 \geq \frac{1}{2}$, but ϕ will suffer a branch cut discontinuity at, say $\theta = \frac{3}{2}\pi$.