231 Tutorial Sheet 20: solution.[1]

8 May 2008

Useful facts:
Questions

1. The function $\phi(x,y)$ is harmonic in the square $0 \le x \le \pi$, $0 \le y \le \pi$. On three sides ϕ is zero and on the lower side

$$\phi(x,0) = \cos x. \tag{1}$$

Determine $\phi(x,y)$ within the square.

*Solution:*Let $\phi(x,y) = X(x)Y(y)$ and substitute in to get

$$X''Y + Y''X = 0 \tag{2}$$

and moving stuff around, this gives

$$\frac{X''}{X} = -\frac{Y''}{Y} \tag{3}$$

and so each side depends on a different independent variable, so they must be equal to a constant

$$\begin{aligned} X'' &= EX \\ Y'' &= EY \end{aligned} \tag{4}$$

so there are three classes of solutions $E = 0$

$$X = Ax + B \tag{5}$$

or $E = k^2$

$$X = Ae^{kx} + Be^{-kx} \tag{6}$$

or $E = -k^2$

$$X = A\sin kx + B\cos kx \tag{7}$$

with similar solutions for Y but, because of the extra minus, with the positive and negative E solutions the other way around. Now, we try to match the boundary conditions. At $x = 0$ and $x = \pi$ we have zero ϕ, hence, for Y non-trivial we must have $X(0) = X(\pi) = 0$, as in the notes, it is not possible to match these boundary conditions for the E positive and $E = 0$ solutions, this leaves the $E = -k^2$, $X(0) = 0$ gives $B = 0$ and $X(\pi) = 0$ give

$$\sin k\pi = 0 \tag{8}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

and hence $k = n$ a positive natural number, negative numbers give the same solutions and $n = 0$ is just the $E = 0$ solution which was dealt with seperately.

Now, our general solution, satisfying the x boundary conditions and the differential equation, is

$$\phi(x, y) = \sum_{n=1}^{\infty} \sin nx \left(C_n e^{ny} + D_n e^{-ny} \right) \tag{9}$$

and putting $\phi(x, \pi) = 0$ gives

$$C_n e^{n\pi} = -D_n e^{-n\pi} \tag{10}$$

Now, to satisfy the $y = 0$ condition

$$\cos x = \sum_{n=1}^{\infty} A_n \sin nx \tag{11}$$

where $A_n = C_n + D_n$ and $0 < x < \pi$. We want to calculate this as a Fourier series, so we have to extend the boundary function as an odd period function: we need a $f(x)$ such that $f(x) = \cos x$ for $x \in (0, \pi)$, $f(-x) = -f(x)$ and $f(x + 2\pi) = f(x)$. Hence, we define

$$f(x) = \begin{cases} \cos x & x \in (0, \pi) \\ -\cos x & x \in (-\pi, 0) \end{cases} \tag{12}$$

and $f(x + 2\pi) = f(x)$ and we expect to be able to find A_n such that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx \tag{13}$$

because it is a sine series for an odd function, and if we do that, we will satisfy the boundary condition because $f(x)$ reduces to the boundary condition along the boundary. By the formula from Fourier analysis

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx dx \tag{14}$$

Now

$$\begin{aligned} \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx dx &= \frac{1}{\pi} \int_0^{\pi} \left(\sin{(n+1)x} - \sin{(n-1)x} \right) dx \\ &= \frac{1}{\pi} \left[\frac{1}{n-1} \cos{(n-1)x} - \frac{1}{n+1} \cos{(n+1)x} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left(\frac{1 + (-1)^n}{n+1} - \frac{1 + (-1)^n}{n-1} \right) \end{aligned} \tag{15}$$

so

$$A_n = -\frac{4}{n^2 - 1} \tag{16}$$

for n even and zero for n odd. Now, solving for C_n and D_n we get

$$\phi(x,y) = \sum \frac{4}{n^2-1} \sin nx \left(\frac{1}{1-e^{2\pi n}} e^{ny} + \frac{1}{1-e^{-2\pi n}} e^{-ny} \right) \tag{17}$$

where the sum is over even values of n.

2. Repeat the problem with the boundary condition $\phi(x,0) = \sin x$ (again ϕ is zero on the other three sides).

   *Solution:*So everything is as before until we get the solution satisfying the x boundary conditions and the differential equation:

$$\phi(x,y) = \sum_{n=1}^{\infty} \sin nx \left(C_n e^{ny} + D_n e^{-ny} \right) \tag{18}$$

   and, again, as before, putting $\phi(x,\pi) = 0$ gives

$$C_n e^{n\pi} = -D_n e^{-n\pi} \tag{19}$$

   Now, to satisfy the $y = 0$ condition we need

$$\sin x = \sum_{n=1}^{\infty} A_n \sin nx \tag{20}$$

   Finding the odd periodic extension of $\sin x$ is easy, it is already odd and periodic, in fact, it is easy to see that the boundary condition is satisfied by $A_1 = 1$ and all others zero, hence

$$\phi(x,y) = \sin x \left(\frac{1}{1-e^{2\pi}} e^{y} + \frac{1}{1-e^{-2\pi}} e^{-y} \right) \tag{21}$$

3. Repeat the problem with the Neumann boundary condition

$$\frac{\partial \phi}{\partial x} = 0 \tag{22}$$

   at $x = 0$ and $x = \pi$,

$$\frac{\partial \phi}{\partial y} = 0 \tag{23}$$

   at $y = \pi$ and

$$\frac{\partial \phi}{\partial y} = -1 \tag{24}$$

   on $y = 0$

*Solution:*So, again, we do the seperation of variable and have three types of solution for each condition; lets consider the boundary conditions at $x = 0$ and π,

$$X'(0) = X'(\pi) = 0 \tag{25}$$

Now, if $E = 0$ and $X = Ax + B$ then $X' = A$ and the conditions tell us that $A = 0$, they say nothing about B, as for the exponential solutions

$$\begin{aligned} X &= Ae^{kx} + Be^{-kx} \\ X' = &= Ake^{kx} - Bke^{-kx} \end{aligned} \tag{26}$$

so $X'(0) = 0$ says that $A = B$ and so $X'(\pi) = 0$ says that there are no non-trivial solutions. That leaves the oscillating solutions

$$\begin{aligned} X &= A \sin kx + B \cos kx \\ X' = &= A \cos kx - B \sin kx \end{aligned} \tag{27}$$

now $X(0) = 0$ says $A = 0$ and $X(\pi) = 0$ says that $k = n$ a natural number, hence

$$\phi(x, y) = \sum_{n=1}^{\infty} \cos nx \left(C_n e^{ny} + D_n e^{-ny}\right) + C + Dy \tag{28}$$

is the general solution. Hence

$$\frac{\partial \phi(x, y)}{\partial y} = \sum_{n=1}^{\infty} \cos nx \left(C_n n e^{ny} - D_n n e^{-ny}\right) + D \tag{29}$$

and setting this to zero at $y = \pi$ gives $D = 0$ and

$$C_n e^{n\pi} = D_n e^{-n\pi} \tag{30}$$

Finally, we have the bottom boundary condition

$$-1 = \sum_{n=1}^{\infty} A_n \cos nx \tag{31}$$

where $A_n = C_n + D_n$. Obviously this has no solution, integrating both sides from zero to π would give π on the left but zero on the right. We are forced to realise that the Neumann[2] does not always have a non-singular solution. In fact the 2-d Neumann problem for the Laplace equation on a region D only has a solution when

$$\int_{\delta D} \nabla \phi \cdot \mathbf{dS} = 0 \tag{32}$$

which it doesn't here.

[2]Named for Carl Neumann a German mathematician of the late 19th, early 20th centuries, not John van Neumann as I may have implied

4. $\phi(x,y)$ is harmonic in the strip $0 \leq y \leq 1$ and periodic in the x direction. On the upper and lower edges Dirichlet boundary conditions are imposed

$$\phi(x, y=1) = 1 + \sin x, \qquad \phi(x, y=0) = \cos 2x.$$

Determine $\phi(x,y)$ within the strip.

Suggestion: When applying the separation of variables method do not forget the case

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = 0. \tag{33}$$

*Solution:*So again we use the seperation of variables and try to impose boundary conditions. Since ϕ is periodic in the x direction we must have $E = 0$ or $E = -k^2$ now, if $E = 0$,

$$\phi = (Ax + B)(Cy + D) \tag{34}$$

so X periodic implies $A = 0$ and $X(0) = \cos 2x$ means that $D = 0$, absorbing a constant this leaves $X = Cy$ so putting $y = 0$ we get $\phi = y$ is a solution $\phi(x, y = 1) = 1$, $\phi(x, y = 0) = 0$, next lets match the bottom boundary condition, $\phi(x, y = 0) = \cos 2x$, this means $n = 2$ and so

$$\phi = \cos 2x \left(Ce^{2y} + De{-}2y\right) \tag{35}$$

with $C + D = 1$ to give the bottom boundary and $Ce^2 + De^{-2} = 0$ to give the bottom boundary; this

$$\phi = \cos 2x \left(\frac{1}{1-e^4}e^{2y} + \frac{1}{1-e^{-4}}e{-}2y\right) \tag{36}$$

satisfies $\phi(x, 0) = cos 2x$ and $\phi(x, 1) = 0$; finally

$$\phi = \sin x \left(Ce^{y} + De{-}y\right) \tag{37}$$

satisfies the top condition provided

$$\begin{aligned} C + D &= 0 \\ Ce + De^{-1} &= 1 \end{aligned} \tag{38}$$

which gives

$$\phi = \sin x \left(-\frac{e}{1-e^2}e^{y} + \frac{e}{1-e^2}e{-}y\right) \tag{39}$$

which satisfies $\phi(x, 0) = 0$ and $\phi(x, 1) = \sin x$. Now, we put them all together to get

$$\phi = y + \cos 2x \left(\frac{1}{1-e^4}e^{2y} + \frac{1}{1-e^{-4}}e{-}2y\right) + \sin x \left(-\frac{e}{1-e^2}e^{y} + \frac{e}{1-e^2}e{-}y\right) \tag{40}$$

which solves the problem.