231 Tutorial Sheet 19: outline solutions.[1]

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Useful facts:

1. Prove uniqueness for solutions of the Klein-Gordon or Helmholz equation

$$\triangle\phi = m^2\phi \tag{1}$$

on a region D and with Dirichlet or Neumann boundary conditions on δD.

*Solution:*So, the way to do this is to again define an energy

$$E = \int_V dV[(\nabla\phi)^2 + m^2\phi^2] \tag{2}$$

and, if there are two solutions ϕ_1 and ϕ_2, let $\phi = \phi_1 - \phi_2$; this will satisfy the pde with zero boundary conditions. As for the Laplace equation case, we take one of the ∇'s outside to give a boundary term

$$E = \int_V dV[(\nabla\phi)^2 + m^2\phi^2] = \int_V dV[\nabla(\phi\nabla\phi) + m^2\phi^2 - \phi\triangle\phi] \tag{3}$$

where we have taken away a $\phi\triangle\phi$ since, from our identities, for ϕ a scalar and $\mathbf{F}$ a vector field

$$\nabla(\phi\mathbf{F}) = \nabla\phi\cdot\mathbf{F} + \phi(\nabla\cdot\mathbf{F}) \tag{4}$$

Now the last two terms cancel because of the equation; if we hadn't added the $m^2\phi^2$ term to the energy this bit would not have worked. Hence

$$E = \int_V dV[\nabla(\phi\nabla\phi) = \int_{\delta V} \phi(\nabla\phi)\cdot\mathbf{dS} = 0 \tag{5}$$

where we have used the Gauss theorem and we get zero because of the boundary conditions. Now E can only be zero if $\phi = 0$.

2. Prove uniqueness for solutions to the heat equation

$$\triangle u = k\frac{\partial u}{\partial t} \tag{6}$$

on a region $D \times [0, \infty)$ and with Dirichlet or Neumann boundary conditions on $\delta D \times [0, \infty)$, initial condition $u(\mathbf{x}, 0) = f(\mathbf{x})$ on D at time $t = 0$ and decay condition $u(\mathbf{x}, t) \to 0$ exponential fast at as t goes to infinity, k is a constant.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

*Solution:*Again, say we have two solutions, u_1 and u_2, then the difference $u = u_1 - u_2$ satisfies the equation with zero boundary conditions. Consider

$$E = \int_0^\infty \int_V dV[(\nabla u)] \tag{7}$$

and again taking one of the ∇ outside

$$\begin{aligned} E &= \int_0^\infty dt \int_V dV[(\nabla u)] \\ &= \int_0^\infty dt \int_V dV[\nabla(u\nabla u) - u\triangle u] \\ &= \int_0^\infty dt \int_V dV[u\frac{\partial}{\partial t}u] \end{aligned} \tag{8}$$

Where again we have used the Gauss theorem and the boundary conditions. The only difference is that getting rid of the Laplace term has left us with a time derivative term, however, this can be rewritten as a total derivative

$$\begin{aligned} E &= \int_0^\infty dt \int_V dV[u\frac{\partial}{\partial t}u] \\ &= \frac{1}{2}\int_V dV \int_0^\infty dt[\frac{\partial}{\partial t}u^2] \\ &= \frac{1}{2}\int_V dV\ u^2\Big]_0^\infty = 0 \end{aligned} \tag{9}$$

so again we have $\nabla u = 0$ which means u is constant in space and substituting into the heat equation tells us it is also constant in time, and, by the initial condition, that constant is zero.