231 Outline Solutions Tutorial Sheet 16, 17 and 18.[1,2]

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Problem Sheet 16

1. Solve $x^2y'' + 4xy' + y = 0$. *Solution:*$x^2y'' + 4xy' + y = 0$. The standard substitution $x = e^z$ gives
$$\frac{d^2y}{d^2z} + (4-1)\frac{dy}{dz} + y = 0. \tag{1}$$
Auxiliary equation $\lambda^2 + 3\lambda + 1 = 0$ with roots $\lambda = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$
$$y = C_1e^{-(\frac{3}{2}-\frac{1}{2}\sqrt{5})z} + C_2e^{-(\frac{3}{2}+\frac{1}{2}\sqrt{5})z} = C_1x^{-(\frac{3}{2}+\frac{1}{2}\sqrt{5})} + C_2x^{-(\frac{3}{2}-\frac{1}{2}\sqrt{5})}. \tag{2}$$

2. Solve $x^2y'' + 4xy' + y = x^5$. *Solution:*So the same substitution gives
$$\frac{d^2y}{d^2z} + 3\frac{dy}{dz} + y = e^{5z} \tag{3}$$
So, we already have the complementary function for this, we just need the particular integral, substitute $y = C\exp(5z)$ giving
$$25C + 15C + C = 1 \tag{4}$$
giving
$$y = C_1x^{-(\frac{3}{2}+\frac{1}{2}\sqrt{5})} + C_2x^{-(\frac{3}{2}-\frac{1}{2}\sqrt{5})} + \frac{1}{41}x^5. \tag{5}$$
This isn't such a good question, the one I meant to ask was something like
$$x^2y'' - 3xy' - 5y = x^5 \tag{6}$$
After substitution this gives
$$\frac{d^2y}{d^2z} - 4\frac{dy}{dz} - 5y = e^{5z} \tag{7}$$
So, the complementary equation is
$$\lambda^2 - 4\lambda - 5 = 0 \tag{8}$$
leading to $\lambda = 5$ or $\lambda = -1$. Now, to get the particular integral, we need to substitute $y = Cz\exp(5z)$. Hence
$$10C - 4C = 1 \tag{9}$$

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[2]Including material from Chris Ford, to whom many thanks.

so $C = 1/6$ and
$$y = C_1x^5 + \frac{C_2}{x} + \frac{1}{6}x^5\log x \tag{10}$$
Another question along the same lines would be
$$x^2y'' + 3xy' + y = 0 \tag{11}$$
Here substituting $x = e^z$ yields
$$\frac{d^2y}{d^2z} + (3-1)\frac{dy}{dz} + y = 0.$$
Auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = -1$ so that $y(x) = C_1e^{-z} + C_2ze^{-z} = C_1x^{-1} + C_2x^{-1}\log x$.

3. Assuming the solution of
$$(1-x)y' + y = 0 \tag{12}$$
has a series expansion about $x = 0$ work out the recursion relation. Write out the first few terms and show that the series $a_2 = 0$ so the series actually terminates to give $y = A(1-x)$ for arbitrary A. What is the solution with $y(0) = 2$.

*Solution:*Well we begin by writing
$$y = \sum_{n=0}^{\infty} a_nx^n \tag{13}$$
and so by differentiation we get
$$y' = \sum_{n=0}^{\infty} a_nnx^{n-1} \tag{14}$$
and hence
$$xy' = \sum_{n=0}^{\infty} a_nnx^n. \tag{15}$$
Thus, substituting the differential equation we get
$$\sum_{n=0}^{\infty} a_nnx^{n-1} - \sum_{n=0}^{\infty} a_nnx^n + \sum_{n=0}^{\infty} a_nx^n = 0 \tag{16}$$
In order to make progress we need to rewrite the first of these three series so that it is in the form
$$\sum_{n=0}^{\infty} \text{stuff}_nx^n \tag{17}$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y', (**??**), to get

$$y' = \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m. \tag{18}$$

In fact, this looks at first like it gives

$$y' = \sum_{m=-1}^{\infty} a_{m+1}(m+1)x^m \tag{19}$$

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n, don't get confused, this isn't the original n, we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{20}$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1) + (1-n)a_n]x^n = 0. \tag{21}$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \tag{22}$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0. \tag{23}$$

For $n = 1$ we get

$$a_2 = 0 \tag{24}$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-x)$ for arbitrary a_0. If $y(0) = 2$ then $a_0 = 2$ and $y = 2(1-x)$.

Problem Sheet 17

1. Use the recursion relation

$$a_{n+2} = \frac{2(n-\alpha)a_n}{(n+1)(n+2)}$$

or the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

to obtain polynomial solutions of Hermite's equation $y'' - 2xy' + 2\alpha y = 0$ for $\alpha = 3$, 4 and 5.

*Solution:*Ok lets use the generating function, so, we want everything up to h^5

$$\begin{aligned} \Phi(x,h) &= e^{2xh-h^2} \\ &= 1 + (2xh - h^2) + \frac{1}{2}(2xh-h^2)^2 + \frac{1}{6}(2xh-h^2)^3 \\ &\quad + \frac{1}{24}(2xh-h^2)^4 + \frac{1}{120}2^5x^5h^5 + O(h^6) \end{aligned} \tag{25}$$

and, continuing to drop high powers in h

$$\begin{aligned} \Phi(x,h) &= 1 + 2xh - h^2 + \frac{1}{2}(4x^2h^2 - 4xh^3 + h^4) + \frac{1}{6}(8x^3h^3 - 12x^2h^4 + 6xh^5) \\ &\quad + \frac{1}{24}(16x^4h^4 - 32x^3h^5) + \frac{1}{120}32x^5h^5 + O(h^6) \end{aligned} \tag{26}$$

Hence

$$\begin{aligned} P_3(x) &= 8x^3 - 12x \\ P_4(x) &= 16x^4 - 48x^2 + 12 \\ P_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned} \tag{27}$$

2. Legendre's equation can be written

$$(1-x^2)y'' - 2xy' + \alpha y = 0,$$

where α is a constant. Consider a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine a recursion relation for the a_n coefficients. For what values of α does Legendre's equation have polynomial solutions?

Solution:$y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ Therefore $xy'(x) = \sum_{n=0}^{\infty} n a_n x^n$ and $x^2y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^n$.

Relabel $y''(x)$ $(n = m+2)$

$$y''(x) = \sum_{m=-2}^{\infty} a_{m+2}(m+2)(m+1)x^m = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m.$$

The last step used that the first two terms in the sum are zero. The ODE can be written

$$\begin{aligned}(1-x^2)y''(x) \quad - \quad & 2xy'(x) + \alpha y(x) \\ &= \sum_{m=0}^{\infty} x^m \left[a_{m+2}(m+2)(m+1) - a_m m(m-1) - 2ma_m + \alpha a_m\right] \\ &= 0, \end{aligned} \tag{28}$$

giving the recursion relation

$$a_{m+2} = \frac{m(m+1)-\alpha}{(m+2)(m+1)} a_m.$$

If α is of the form $n(n+1)$ $(n = 0, 1, 2, ...)$ *one* of the solutions of the ODE will be a polynomial since the recursion relation will terminate.

3. (Frobenius training exercise) For each of the following equations obtain the indicial equation for a Frobenius series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

(a) $y'' + y = 0.$

(b) $x^2 y'' + 3xy' + y = 0$

(c) $4xy'' + 2y' + y = 0.$

In case a) use the method of Frobenius to obtain the general solution. In case b) use the method of Frobenius to find one solution (the method fails to give the other solution).

Solution:(a) Frobenius: $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ where s is to be determined. Differentiating twice gives $y''(x) = \sum (n+s)(n+s-1)x^{n+s-2}$ (most singular term in ODE). Relabel this as $y''(x) = a_0 s(s-1)x^{s-2} + a_1(s+1)sx^{s-1} + \sum_{m=0}^{\infty} a_{m+2}(m+s+2)(m+s+1)x^{m+s}$ and so

$$\begin{aligned} y''(x) + y(x) &= a_0 s(s-1)x^{s-2} + a_1(s+1)sx^{s-1} \\ &+ \sum_{m=0}^{\infty} x^{m+s}\left[a_{m+2}(m+s+2)(m+s+1) + a_m\right] = 0. \end{aligned}$$

Take $a_0 = 1$. The indicial equation is

$$s(s-1) = 0,$$

with roots $s = 0$ and $s = 1$.

$\underline{s=0}$ For this s can take $a_1 \neq 0$ but for now set $a_1 = 0$.

Recursion relation: $a_{n+2}(n+2)(n+1) + a_n = 0$ or

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)},$$

which is particularly easy to solve $a_0 = 1$, $a_2 = -\frac{1}{1\cdot 2}$, $a_4 = \frac{1}{1\cdot 2\cdot 3\cdot 4}$, ..., $a_{2p} = \frac{(-1)^p}{(2p)!}$. The solution is

$$y(x) = \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{(2p)!} = \cos x.$$

Including $a_1 \neq 0$ gives $y(x) = \cos x + a_1 \sin x$. The $s = 1$ solution is also $y(x) = \sin x$. The general solution is

$$y(x) = A\cos x + B \sin x.$$

(b) $x^2y'' + 3xy' + y = 0$. Frobenius $y = \sum_{n=0} a_n x^{n+s}$, $xy' = \sum_{n=0} a_n (n+s)x^{n+s}$, $x^2 y'' = \sum_{n=0} a_n (n+s)(n+s-1)x^{n+s}$. No 'most singular' term or terms! No recursion relation! Indicial equation: need a_0 contributions, $x^2y'' + 3xy' + y = a_0 x^s\left[s(s-1) + 3s + 1\right]$ + higher powers $= 0$ so that $s^2 + 2s + 1 = 0$ with two equal roots $s = -1$. Since there is no recursion relation the a_n $(n > 0)$ are all zero. $y(x) = x^{-1}$ is one solution, the other solution is not a Frobenius series.

(c) Write $y = \sum_{n=0}^{\infty} \; a_n \; x^{n+s}$

$y' = \sum_{n=0}^{\infty} \; a_n \; (n+s) \; x^{n+s}, \qquad y'' = \sum_{n=0}^{\infty} \; a_n \; (n+s)(n+s-1) \; x^{n+s}$. Now:

$y'(x) = a_0 s x^{s-1} + \sum_{n=0}^{\infty} a_{n+1}(n+1+s)x^{n+s}$

$xy''(x) = a_0 s(s-1)x^{s-1} + \sum_{m=0}^{\infty} a_{m+1}(m+1+s)(m+s)x^{m+s}$.

$$\begin{aligned} 4xy'' + 2y' + y \quad = \quad & a_0\left[4s(s-1) + 2s\right] \\ & + \sum_{m=0}^{\infty}\left[4(m+1+s)(m+s)a_{m+1} + 2(m+1+s)a_{m+1} + a_m\right]x^{m+s} \\ & 4a_0 s(s-\frac{1}{2})x^{s-1} + \sum_{m=0}^{\infty}\left[4(m+1+s)(m+s+\frac{1}{2})a_{m+1} + a_m\right]. \end{aligned}$$

Set $a_0 = 1$ Indicial equation: $s(s-\frac{1}{2}) = 0$ with roots $s = 0$ and $s = \frac{1}{2}$.

4. Use the recursion relation to show that the functions H_n defined through the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

satisfy Hermites equation

$$y'' - 2xy' - 2ny = 0.$$

*Solution:*So

$$\frac{\partial}{\partial x}\Phi(x,h) = 2he^{2xh-h^2} = \sum_{n=0}^{\infty}\frac{h^n}{n!}H_n'(x) \tag{29}$$

and

$$\frac{\partial^2}{\partial x^2}\Phi(x,h) = 4h^2e^{2xh-h^2} = \sum_{n=0}^{\infty}\frac{h^n}{n!}H_n''(x) \tag{30}$$

So

$$\frac{\partial^2}{\partial x^2}\Phi(x,h) - 2x\frac{\partial^2}{\partial x^2}\Phi(x,h) = 4(h-x^2)he^{2xh-h^2} \tag{31}$$

The trick now is to spot[3] that the right hand side is

$$-4(h-x^2)he^{2xh-h^2} = 2h\frac{\partial}{\partial h}e^{2xh-h^2} = \sum_{n=0}^{\infty}n\frac{h^n}{n!}H_n(x) \tag{32}$$

Hence

$$\frac{\partial^2}{\partial x^2}\Phi - 2x\frac{\partial^2}{\partial x^2}\Phi + 2h\frac{\partial}{\partial h}\Phi = 0 \tag{33}$$

and writing this out in terms of the sums gives

$$\sum_{n=0}^{\infty}\frac{h^n}{n!}\left[H_n''(x) - 2xH_n'(x) + 2nH(x)\right] \tag{34}$$

and equating each coefficient of each power of h to zero gives the result.

Doing the same question using the recursion relation is more complicated, you need to consider the expansion:

$$e^{2xh-h^2} = \sum_{n=0}^{\infty}\frac{h^n(2x-h)^n}{n!} = \sum_{n=0}^{\infty}\frac{h^n}{n!}\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}(2x)^{n-m}(-1)^m h^m \tag{35}$$

where the last expression comes from the binomial expansion. Now, we just need to do a change of index to get this into the form

$$e^{2xh-h^2} = \sum_{n=0}^{\infty}\frac{h^n}{n!}H_n(x) \tag{36}$$

Lets start by setting $p = m + n$, the current index of h, the complication here is that n appears in the sum range of m, so the end point of the m sum is $m = p - m$; it is good to check by hand how the sums ranges change, but basically this means $m = p/2$ for p even and $m = (p-1)/2$ for m odd.

$$e^{2xh-h^2} = \sum_{p=0}^{\infty}\sum_{m=0}^{M(p)}\frac{1}{(p-m)!}\begin{pmatrix} p-m \\ m \end{pmatrix}(2x)^{p-2m}(-1)^m h^p \tag{37}$$

[3] I have now noticed that when Chris set a question like this a few years ago he gave a hint like this

where $M(p)$ denotes the correct end point for even and odd p. Now, we mess around a bit: lets concentrate on the even sum and let $p = 2q$ and $j = q - m$, so

$$e^{2xh-h^2} = \text{odd} + \sum_{q=0}^{\infty}\sum_{j=0}^{q}\frac{1}{(2j)!(q-j)!}2^{2j}x^{2j}(-1)^j h^{2q} \tag{38}$$

and hence

$$H_{2q}(x) = \sum_{j=0}^{q}\frac{(2q)!}{(2j)!(q-j)!}2^{2j}x^{2j}(-1)^j \tag{39}$$

giving

$$a_{2j} = \frac{(2q)!}{(2j)!(q-j)!}2^{2j}(-1)^j \tag{40}$$

and

$$a_{2j+2} = \frac{(2q)!}{(2j+2)!(q-j-1)!}2^{2j+2}(-1)^{j+1} \tag{41}$$

$$= -4\frac{q-j}{(2j+2)(2j+1)}a_{2j} = 2\frac{2j-2q}{(2j+2)(2j+1)}a_{2j} \tag{42}$$

which, since $\alpha = 2q$ and n in the notes is our $2j$, after all these changes of index, is what we want!

Problem Sheet 18

1. Bessel's equation reads

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

In the lectures it was shown that inserting a Frobenius series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

with $a_0 \neq 0$ leads to $a_1 = 0$, the indical equation, $s^2 - \nu^2 = 0$, and the recursion relation

$$a_{n+2} = -\frac{a_n}{(n+s+2)^2 - \nu^2}.$$

For $\nu = 0$ this leads to two equal roots $s = 0$ and so the method only provides one solution. Use the recursion relation to compute the a_n for this case.

*Solution:*So this is just a question of applying the recursion relation with $\nu = 0$, keeping in mind that $s = \nu$ in this case:

$$a_{n+2} = -\frac{a_n}{(n+2)^2}. \tag{43}$$

Now, $a_0 \neq 0$ and $a_1 = 0$; for convenience, let $a_0 = 1$, we can always multiply by an overall arbitrary constant afterwards; $n = 0$ gives

$$a_2 = -\frac{1}{4} \tag{44}$$

and $n = 2$

$$a_4 = -\frac{1}{16}a_2 = \frac{1}{16 \times 4} \tag{45}$$

and $n = 6$

$$a_6 = -\frac{1}{36}a_4 = \frac{1}{36 \times 16 \times 4}. \tag{46}$$

Hence, it is clear

$$a_{2m} = (-1)^m \frac{1}{2^{2m}(m!)^2} \tag{47}$$

So we now have

$$y = C\sum_{m=0}^{\infty}(-1)^m \frac{x^{2m}}{2^{2m}(m!)^2} \tag{48}$$

2. Use the method of Frobenius to obtain the general solution to the ODE

$$4xy''(x) + 2y'(x) + y(x) = 0.$$

*Solution:*This question is actually in problem sheet 17 above.

3. In each of the following cases find a second solution in the form $y(x) = u(x)v(x)$ where $u(x)$ is a solution and $v(x)$ is to be determined.

(a) $y'' + 5y' + 6y = 0$; find one solution using the auxiliary and the other using the $y = uv$ ansatz.

(b) $(1 - x^2)y'' - 2xy' = 0$ one solution is $u(x) = 1$.

Remark: Part b) is the $\alpha = 0$ case of Legendre's equation.

*Solution:*For the first one, use the auxhillary equation to find that one solution of $y'' + 5y' + 6y = 0$ is $u(x) = e^{-2x}$. Write $y(x) = u(x)v(x) = e^{-2x}v(x)$. $y'(x) = e^{-2x}v'(x) - 2e^{-2x}v(x)$, $y''(x) = e^{-2x}v''(x) - 4e^{-2x}v'(x) + 4e^{-2x}v(x)$ so the ODE becomes $y''(x) + 5y'(x) + 6y(x) = e^{-2x}\left(v''(x) + (5-4)v'(x)\right) = 0$ or $v''(x) + v'(x) = 0$ (a first order ODE for v') with solution $v'(x) = Ce^{-x}$. Integrating $v(x) = C_1e^{-x} + C_2$, relabel $C_1 = -C$, so a second solution of the ODE is $y(x) = e^{-3x}$.

Now, for the second part, $(1-x^2)y'' - 2xy' = 0$, $u(x) = 1$. Taking $y(x) = u(x)v(x) = v(x)$ gives $(1 - x^2)v'' - 2xv' = 0$ a first order ODE for v', i.e.

$$\frac{dv'}{v'} = \frac{2x\ dx}{1 - x^2}$$

Integrating

$$\log v' = -\log(1 - x^2) + c,$$

or,

$$v' = \frac{C}{(1 - x^2)} = \frac{C}{2}\left(\frac{1}{1 - x} + \frac{1}{1 + x}\right).$$

Integrate again, relabel $C_1 = \frac{1}{2}C$,:

$$v = C_1 \log\left(\frac{1 + x}{1 - x}\right) + C_2.$$

A second solution is thus $y(x) = \log\frac{1+x}{1-x}$.

4. The Legendre polynomials $P_n(x)$ are generated by

$$\Phi(x, h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{49}$$

Write down the first four Legendre polynomials and verify that they are orthogonal

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \tag{50}$$

for $n \neq m$.

*Solution:*So first we need to work out the polynomials, recall the binomial expansion

$$\frac{1}{\sqrt{1 - \epsilon}} = 1 + \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \frac{5}{16}\epsilon^3 + O(epsilon^4) \tag{51}$$

hence

$$\begin{aligned}
\Phi(x, h) &= \frac{1}{\sqrt{1 - 2xh + h^2}} \\
&= 1 + \frac{1}{2}(2xh - h^2) + \frac{3}{8}(2xh - h^2)^2 + \frac{5}{16}(2xh - h^2)^3 + O(h^4) \\
&= 1 + \frac{1}{2}(2xh - h^2) + \frac{3}{8}(4x^2h^2 - 4xh^3) + \frac{5}{16}8x^3h^3 + O(h^4) \qquad (52)
\end{aligned}$$

Therefore, we can read off the Legendre polynomials

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\
P_3(x) &= -\frac{3}{2}x + \frac{5}{2}x^3 \qquad (53)
\end{aligned}$$

Finally, we need to check that these are all orthogonal; mixing even and odd will give an odd polynomial, which will integrate to zero over $[-1,1]$; so

$$\int_{-1}^{1} P_0 P_2 \, dx = \frac{1}{2} \int_{-1}^{1} (3x^2 - 1) dx = 0 \tag{54}$$

and

$$\int_{-1}^{1} P_1 P_3 \, dx = \frac{1}{2} \int_{-1}^{1} (-3x^2 + 5x^4) dx = 0 \tag{55}$$