231 Outline Solutions Tutorial Sheet 13, 14 and 15.[1,2]

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Problem Sheet 13

1. Express the following functions as Fourier integrals:

(a)

$$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

(b)

$$f(x) = \frac{\sin x}{x}$$

Solution:(a) Writing f as a Fourier integral $f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k)$. We require the Fourier transform:

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{-ikx}\ f(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk\ e^{-ikx}\ \frac{e^{ix}+e^{-ix}}{2} \\
&= \frac{1}{4\pi}\left(\frac{e^{i(1-k)x}}{i(1-k)} + \frac{e^{i(-1-k)x}}{i(-1-k)}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-ik\pi/2}+ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2}-ie^{ik\pi/2}}{i(-1-k)}\right] \\
&= \frac{1}{4\pi}\ 2\cos\left(\frac{k\pi}{2}\right)\ \left(\frac{1}{1-k}+\frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right)\ \frac{1}{1-k^2}.
\end{aligned}$$

Therefore

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\ \cos\left(\frac{k\pi}{2}\right)\ \frac{e^{ikx}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$. (b)

$$\frac{\sin x}{x} = \frac{1}{2}\int_{-1}^{1} dk\ e^{ikx}.$$

Remark: In the lectures it was shown that the Fourier transform of a square pulse is proportional to $\sin k/k$ and so it follows that the Fourier transform of the $\sin x/x$ is proportional to the pulse and, for example, integrating quickly gives the constant of proportionality.

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[2]Including material from Chris Ford, to whom many thanks.

2. Prove the following properties of the Fourier transform

   (a) The Fourier transform of an even function is even.

   (b) $\tilde{f'}(k) = ik\tilde{f}(k)$.

   Solution:(a) Assume that f is even, i.e. $f(-x) = f(x)$, then

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikx} f(x).$$

   make the change of variables $y = -x$:

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{-iky} f(-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{-iky} f(y) = \tilde{f}(k).$$

   (b) here an integration by parts is required

$$\tilde{f'}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-ikx} f'(x) = e^{-ikx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx(-ik)e^{-ikx} f(x) = ik\tilde{f}(k),$$

   assuming that the boundary terms vanish.

3. In the lectures (quite a while ago) it was shown that the scalar field

$$\phi(\mathbf{r}) = \frac{1}{r},$$

   where $r = \sqrt{x^2+y^2+z^2}$ is harmonic except at the origin. In fact it can be shown that

$$\nabla^2\phi(\mathbf{r}) = -4\pi\delta^3(\mathbf{r}). \qquad (A)$$

   Formally apply Gauss' theorem to the vector field $\mathbf{F} = \nabla\phi$ to show that

$$\int_{r<a} dV\ \nabla^2\ \phi = -4\pi.$$

   This is clearly consistent with (A). Another treatment would replace the singular scalar field ϕ with a sequence of smooth scalar fields, e.g.

$$\phi_n(\mathbf{r}) = \frac{n}{\sqrt{n^2r^2+1}}.$$

   Prove that

$$\int_{R^3} dV\ \nabla^2\phi_n(\mathbf{r}) = -4\pi.$$

   Solution:$\mathbf{F} = \nabla\phi$ so that $\nabla^2\phi =$ div $\mathbf{F}$. Applying Gauss' theorem

$$\int_{r<a} dV\ \nabla^2\phi = \int_{r<a} dV\ \text{div}\ \mathbf{F} = \int_{r=a} \mathbf{F}\cdot\mathbf{dA}.$$

$\mathbf{F} = -\mathbf{r}/r^3$ and $\mathbf{F}\cdot\mathbf{n} = -1/a^2$ and the surface are is $4\pi a^2$ giving

$$\int_{r<a} dV\ \nabla^2\phi = -4\pi.$$

$\partial_x\phi_n = -\frac{1}{2}n(n^2r^2+1)^{-3/2}2xn^2$, and similarly for $\partial_y\phi_n$ and $\partial_z\phi_n$. Therefore

$$\nabla\phi_n = -\frac{n^3\mathbf{r}}{(n^2r^2+1)^{3/2}}.$$

$$\int_{r<a} dV\ \nabla^2\phi_n = \int_{r=a} \nabla\phi_n\cdot\mathbf{dA} = -\frac{n^3\ 4\pi a^3}{(n^2a^2+1)^{3/2}} \to -4\pi$$

as $a\to\infty$.

Problem Sheet 14

1. Inside an integral, what is

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} \tag{1}$$

for $\theta(x)$ the usual Heaviside function and $|\epsilon|<1$.

Solution: There are two ways to do this, we can either expand the fraction as a power series or we can try and evaluate it inside an integral. First the first way, using the usual expansion of $1/(1+x)$ for $x<1$

$$\frac{1}{1+\epsilon\theta(x)} = \sum_{n=0}^{\infty}[-\epsilon\theta(x)]^n \tag{2}$$

Now, we just use the fact that $\theta(x)^n = \theta(x)$ for n a positive integer; note that we have to be careful with the first term in the series which doesn't contain a $\theta(x)$ factor since n is zero. Hence

$$\frac{1}{1+\epsilon\theta(x)} = 1+\theta(x)\sum_{n=1}^{\infty}(-\epsilon)^n = 1+\theta(x)\sum_{n=0}^{\infty}(-\epsilon)^n - \theta(x) = 1-\theta(x)+\frac{1}{1+\epsilon}\theta(x) \tag{3}$$

where we have added and taken away the missing term in the sum. Hence,

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} = \frac{d}{dx}[1-\theta(x)+\frac{1}{1+\epsilon}\theta(x)] = \frac{1}{1+\epsilon}\delta(x)-\delta(x) \tag{4}$$

The other way is to do the calculation inside an integral: with $a<0$ and $b>0$

$$I = \int_a^b dx f(x)\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} \tag{5}$$

Now, integrating by parts

$$I = f'(x)\frac{1}{1+\epsilon\theta(x)}\Bigg]_a^b - \int_a^b dx f'(x)\frac{1}{1+\epsilon\theta(x)} \tag{6}$$

Next, we split the integral into two and use the fact $\theta(x)$ is zero for negative x and one for positive x. So,

$$I = \frac{1}{1+\epsilon}f(b) - f(a) - \int_a^0 dx f'(x) - \frac{1}{1+\epsilon}\int_0^b dx f'(x) \tag{7}$$

and, using the Fundamental Theorem of Calculus

$$I = \frac{1}{1+\epsilon}f(b) - f(a) - f(0) + f(a) - \frac{1}{1+\epsilon}f(b) + \frac{1}{1+\epsilon}f(0) = \left(\frac{1}{1+\epsilon} - 1\right)f(0) \tag{8}$$

which implies

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} = \frac{1}{1+\epsilon}\delta(x) - \delta(x) \tag{9}$$

as before.

2. Compute
   (a) $\int_{-\infty}^{\infty} dx\ e^x\ \delta(x+1)$
   (b) $\int_{-3}^{1} dx\ \delta(x^2-3x+2)$
   (c) $\int_{-\infty}^{\infty} dx\ \cos x\ \delta'(x)$
   (d) $\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right)$.

   Solution:

   (a) $\int_{-\infty}^{\infty} dx\ e^x\ \delta(x+1) = e^{-1}$.
   (b) Use

   $$\delta\left(h(x)\right) = \sum_i \frac{\delta(x-x_i)}{|h'(x_i)|},$$

   where the x_is are roots of h. In this case $h(x) = x^2 - 3x + 2 = (x-2)(x-1)$ with roots $x_1 = 2$ and $x_2 = 1$. This is a problem since $x = 1$ is one of the limits of integration, in fact

   $$\int_{-\infty}^{0} dx\delta(x) \tag{10}$$

   isn't defined, and so the answer here is that the integral isn't defined. Say instead we had been asked

   $$\int_{-3}^{3} dx\ \delta(x^2-3x+2) \tag{11}$$

then both roots are in the integral and we would use $h'(x) = 2x - 3$ so that $h'(1) = -1$, giving $|h(1)| = 1$ and $h'(2) = 1$ which gives

$$\delta(x^2 - 3x + 2) = \delta(x + 1) + \delta(x_2)$$

and

$$\int_{-3}^{3} dx\ \delta(x^2 - 3x + 2) = 2.$$

(c)

$$\int_{-\infty}^{\infty} dx\ \cos x \delta'(x) = -\int_{-\infty}^{\infty} dx\ (-\sin x)\ \delta(x) = 0$$

Integrating by parts and using $\sin 0 = 0$.

(d) Use formula for $\delta(h(x))$, here $h(x) = \sin(1/x)$ which is zero for $1/x = n\pi$ $(n \in Z)$. $h'(x) = -x^{-2}\cos(1/x)$ and since $|\cos n\pi| = 1$

$$\delta\left(h(x)\right) = \sum_{n\neq 0} \frac{\delta\left(x - \frac{1}{n\pi}\right)}{\pi^2 n^2}.$$

Now $1/(n\pi) \in (0, 1)$ for all positive n which gives

$$\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right) = \frac{1}{\pi^2}\sum_{n>0}\frac{1}{n^2}.$$

The sum on the RHS is $\zeta(2) = \pi^2/6$ (see Q3 Sheet 11) and so

$$\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right) = \frac{1}{6}.$$

3. Obtain a general solution to

(a) $y' - 3y = e^{-x}$

(b) $y' + y\cot x = \cos x$

(c) $(x + 1)y' + y = (x + 1)^2$

Solution:

(a) Rewrite as

$$e^{-3x}y' - 3ye^{-3x} = e^{-4x}$$

or

$$\left(e^{-3x}y\right)' = e^{-4x}$$

and then integrate.

(b) the quickest thing to do is multiply across by the sine

$$\sin xy' + cosxy = \sin x \cos x \tag{12}$$

and rewritting

$$(\sin xy)' = (\sin^2 x)' \tag{13}$$

hence

$$\sin xy = \sin^2 x + C \tag{14}$$

or

$$y = \sin x + C\mathrm{cosec}\, x \tag{15}$$

(c) $(x+1)y' + y = (x+1)^2$ can again be rewritten

$$[(x+1)y]' = x^2 + 2x + 1 \tag{16}$$

so

$$(x+1)y = \frac{1}{3}x^3 + x^2 + x + C \tag{17}$$

or

$$3y = \frac{x^3 + 3x^2 + 3x + 1}{x+1} + \frac{C}{x+1} = (x+1)^2 + \frac{C}{x+1} \tag{18}$$

with a redefinition of C to get the nice devision at the end, another way to do this would have been to change variables to $z = x + 1$ at the start.

4. Obtain the general solutions of the following ODEs:

(a) $y'' + 5y' + 6y = 0$

(b) $y'' - 2y' + y = 0$

Solution:

(a) $y'' + 5y' + 6y = 0$ so substitute $e^{\lambda x}$ to get the auxiliary equation

$$\lambda^2 + 5\lambda + 6 = 0 \tag{19}$$

so $\lambda = -2$ and $\lambda = -3$ giving solution

$$y = C_1 e^{-2x} + C_2 e^{-3x} \tag{20}$$

(b) $y'' - 2y' + y = 0$ gives auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0 \tag{21}$$

which has $\lambda = 1$ as a repeated root, so

$$y = C_1 e^x + C_2 x e^x \tag{22}$$

Problem Sheet 15

1. Obtain the general solutions of the ODEs

   (a) $y'' + 3y' - 4y = e^{-x}$

   (b) $y'' + 3y' - 4y = e^{-4x}$

   (c) $y'' + 3y' - 4y = \sinh x$

   Solution: Auxiliary equation gives $\lambda^2 + 3\lambda - 4 = 0$ roots $\lambda = -4$, $\lambda = 1$ so we get

   $$y(x) = C_1 e^{-4x} + C_2 e^x \tag{23}$$

   solving the corresponding homogeneous equation

   (a) Now substitute $y = Ce^{-x}$ to get

   $$C - 3C - 4C = 1 \tag{24}$$

   so $C = -1/6$ and

   $$y(x) = C_1 e^{-4x} + C_2 e^x - \frac{1}{6} e^{-x} \tag{25}$$

   (b) This time the right hand side matches one of the complementary solutions, so we substitute $y = Cx \exp(-4x)$, the terms with xs outside the exponential all cancel and we get

   $$-8C + 3C = 1 \tag{26}$$

   or $C = -1/5$ giving

   $$y(x) = C_1 e^{-4x} + C_2 e^x - \frac{1}{5} x e^{-4x} \tag{27}$$

   (c) The particular integral is

   $$y'' + 3y' - 4y = \sinh x = \frac{1}{2}(e^x - e^{-x}) \tag{28}$$

   and so we split it into two parts. The first is $y'' + 3y' - 4y = \frac{1}{2}e^x$, let $y = Cxe^x$ giving $C(2+3) = \frac{1}{2}$ and so $C = \frac{1}{10}$. The second part is $y'' + 3y' - 4y = -\frac{1}{2}e^{-x}$, $y = Ce^{-x}$ so that $C(1-3-4) = -\frac{1}{2}$, hence $C = 1/12$. The solution is therefore

   $$y = C_1 e^{-4x} + C_2 e^x + \frac{1}{10} x e^x + \frac{1}{12} e^{-x} \tag{29}$$

2. Obtain the general solution of the ODE

$$y''(x) + 3y'(x) + 2y(x) = f(x)$$

where f is the periodic function defined by

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

where $a \in (0, \pi)$ is a constant and $f(x + 2\pi) = f(x)$.

Solution: First the complementary function, the auxiliary equation $\lambda^2+3\lambda+2 = 0$ has roots $\lambda = -1$ and $\lambda = -2$. The right hand side is not in the form of an exponential, but can be expanded as a Fourier series

$$f(x) = \frac{a}{pi} + \sum_{n\neq 0} \frac{\sin an}{\pi n} e^{inx} \tag{30}$$

so we want to solve

$$y''(x) + 3y'(x) + 2y(x) = \frac{\sin an}{\pi n} e^{inx} \tag{31}$$

Letting

$$y = Ce^{inx} \tag{32}$$

gives $(-n^2 + 3in + 2)C = \sin an/\pi n$. The constant, $n = 0$ case is solve by $y = a/2\pi$ so

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{a}{2\pi} + \sum_{n\neq 0} \frac{1}{-n^2 + 3in + 2} \frac{\sin an}{\pi n} e^{inx} \tag{33}$$

3. Obtain the general solutions of the ODEs

   (a) $y'' + y = f(x)$, where f is the periodic square wave defined by

   $$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \quad \text{and } f(x + 2\pi) = f(x)$$

   (b) $y'' + y' + 3y = e^{-|x|}$.

   Solution:

   (a) $y''(x) + y(x) = f(x)$. CF: $y_c = A \sin x + B \cos x$. To find the PI write $f(x)$ as a Fourier series (obtained in the lectures)

   $$f(x) = \frac{4}{\pi} \sum_{n>0,\ n \text{ odd}} \frac{1}{n} \sin nx = \frac{2}{\pi i} \sum_{n \text{ odd}} \frac{e^{inx}}{n}.$$

First find a PI for $y'' + y = e^{inx}$. Trying $y = Ce^{inx}$ gives $(-n^2+1)C = 1$ so that $C = 1/(1-n^2)$ unless $n = \pm 1$.

$n = 1$: Try $y = Cxe^{ix}$ so that $y'' = (-xe^{ix} + 2ie^{ix})C$. Therefore a particular solution to $y'' + y = e^{ix}$ is $y = -\frac{1}{2}ixe^{ix}$.

$n = -1$: Similarly particular solution to $y'' + y = e^{-ix}$ is $\frac{1}{2}ixe^{-ix}$

PI for full problem

$$\begin{aligned} y_p &= \frac{2}{\pi i} \sum_{n \text{ odd},\, n \neq \pm 1} \frac{e^{inx}}{n(1-n^2)} + \frac{2}{\pi i} \cdot -\frac{1}{2}ixe^{ix} - \frac{2}{\pi i} \cdot \frac{1}{2}ixe^{-ix} \\ &= \frac{4}{\pi} \sum_{n > 2 \text{ odd}} \frac{\sin nx}{n(1-n^2)} - \frac{2x}{\pi} \cos x. \end{aligned}$$

(b) Complimentary function: $y'' + y' + 3y = 0$. Auxiliary equation: $\lambda^2 + \lambda + 3 = 0$ with roots $\lambda = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{11}$.

$$y = e^{-\frac{1}{2}x} \left(A\cos(\frac{1}{2}\sqrt{11}x) + B\sin(\frac{1}{2}\sqrt{11}x) \right).$$

PI: write $f(x) = e^{-|x|}$ as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk\; e^{ikx}\; \tilde{f}(k) = \int_{-\infty}^{\infty} dk\; e^{ikx} \frac{1}{\pi} \frac{1}{1+k^2}.$$

PI for $y'' + y' + 3y = e^{ikx}$. Trying $y = Ce^{ikx}$ gives $C(-k^2 + ik + 3)e^{ikx} = e^{ikx}$ so that $C = 1/(-k^2 + ik + 3)$. Therefore a PI to the full problem is

$$y_p(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk\; e^{ikx} \frac{1}{1+k^2} \frac{1}{(-k^2 + ik + 3)}.$$