231 Outline Solutions Tutorial Sheet 1, 2 and 3.[1,2]

9 November 2007

Problem Sheet 1

1. Rewrite the integral

$$I = \int_0^1 dx \int_1^{e^x} dy\ \phi(x,y) \tag{1}$$

as a double integral with the opposite order of integration.

*Solution:*The range of y values: $1 \le y \le e$. For a fixed y, x has the range $\log y \le x \le 1$. Hence

$$I = \int_1^e dy \int_{\log y}^1 dx\ \phi(x,y). \tag{2}$$

2. Evaluate

$$I = \int_D dxdyxe^{xy} \tag{3}$$

where D is given by $0 < x < 1$ and $2 < y < 4$.

*Solution:*So rewriting as an iterated integral

$$I = \int_D dxdyxe^{xy} = \int_0^1 dx \int_2^4 dyxe^{xy} \tag{4}$$

and integrating from the middle

$$\int_0^1 dx \int_2^4 dyxe^{xy} = \int_0^1 dx \left(x\frac{1}{x}e^{xy}\right)_2^4 = \int_0^1 dx \left(e^{4x} - e^{2x}\right) = \frac{1}{4}e^4 - \frac{1}{2}e^2 + \frac{1}{4} \tag{5}$$

Here we cunningly made the integration easier by doing the y integration first, in fact is shouldn't make any difference to the answer if the integration is done in the other order, it is definately harder thought:

$$I = \int_D dxdyxe^{xy} = \int_2^4 dy \int_0^1 dxxe^{xy} = \int_2^4 dy \left(\frac{1}{y^2} - \frac{1}{y^2}e^y + \frac{1}{y}e^y\right) \tag{6}$$

where we did the x integral using integration by parts, Now integrating by parts

$$\int_2^4 dy\frac{1}{y}e^y = \frac{1}{y}e^y]_2^4 + \int_2^4 dy\frac{1}{y^2}e^y \tag{7}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

so, substituting this in and cancelling

$$\int_2^4 dy \left(\frac{1}{y^2} - \frac{1}{y^2}e^y + \frac{1}{y}e^y\right) = \frac{1}{y}e^y]_2^4 + \int_2^4 dy \left(\frac{1}{y^2}\right) = \frac{1}{4}e^4 - \frac{1}{2}e^2 + \frac{1}{4} \tag{8}$$

as before.

3. Evaluate

$$I = \int_D dxdy(x+y) \tag{9}$$

where D is given by $0 < y < 1$ and $2y < x < 2$.

*Solution:*So, write as an iterated integral

$$I = \int_D dxdy(x+y) = \int_0^1 dy \int_{2y}^2 dx(x+y) \tag{10}$$

and integrate from the inside out

$$\begin{aligned} \int_0^1 dy \int_{2y}^2 dx(x+y) &= \int_0^1 dy \left(\frac{1}{2}x^2 + xy\right)_{2y}^2 = \int_0^1 dy \left(2 + 2y - 4y^2\right) \\ &= 3 - \frac{4}{3} = \frac{5}{3} \end{aligned} \tag{11}$$

4. Change the order of integration of

$$I = \int_0^{1/2} dy \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dxy \tag{12}$$

and evaluate.

*Solution:*So

$$x = \pm\sqrt{1-4y^2} \tag{13}$$

implies

$$y = \pm\frac{1}{2}\sqrt{1-x^2} \tag{14}$$

and it is easy to see from drawing a picture that

$$I = \int_0^{1/2} dy \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dxy = \int_{-1}^1 dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} dyy \tag{15}$$

Now, integrating we get

$$\int_{-1}^1 dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} dyy = \frac{1}{8}\int_{-1}^1 dx(1-x^2) = \frac{1}{8}\left(2 - \frac{2}{3}\right) = \frac{1}{6} \tag{16}$$

Problem Sheet 2

1. Consider the integral
$$I = \int_D dV\ \phi \tag{17}$$
where D is the interior of the ellipsoid defined by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{18}$$
Write down I as an iterated triple integral.

*Solution:*Upper surface of ellipsoid ia
$$z = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \tag{19}$$
whereas the lower surface is
$$z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{20}$$
The surfaces join at $z = 0$ where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, this provides range of x and y integrations: $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ to $y = +b\sqrt{1 - \frac{x^2}{a^2}}$ and $x = -a$ to $x = a$:
$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ \phi(x, y, z). \tag{21}$$

2. The Gaussian integral formula
$$\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi} \tag{22}$$
can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over R^2:
$$\left(\int_{-\infty}^{\infty} dx\ e^{-x^2}\right)^2 = \int_{R^2} dA\ e^{-x^2-y^2}. \tag{23}$$
By changing to polar coordinates evaluate this integral.

*Solution:*After changing to polars and making sure to include the Jacobian $J = r$
$$\int_{R^2} dA\ e^{-x^2-y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\ re^{-r^2} \tag{24}$$
and then do this integral by substituting $u = r^2$ so $du = 2rdr$ to give
$$I^2 = \pi \int_0^{\infty} due^{-u} = \pi \tag{25}$$
as required.

3. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates
$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv. \tag{26}$$
*Solution:*Well
$$\begin{aligned} J &= \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\| \\ &= \left\| \begin{matrix} u & -v \\ v & u \end{matrix} \right\| \\ &= u^2 + v^2. \end{aligned} \tag{27}$$

4. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*

*Solution:*Range of integration: $z = 0$ to $z = 4 - y$, $y = -\sqrt{4 - x^2}$ to $y = +\sqrt{4 - x^2}$ and $x = -2$ to $x = 2$. Thus the volume is
$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_0^{4-y} dz\ 1 = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy\ (4 - y), \tag{28}$$
the z integral being trivial. The y integral is also straightforward:
$$V = \int_{-2}^{2} dx\ 8\sqrt{4 - x^2} = 8 \cdot 2\pi = 16\pi. \tag{29}$$
The final integral can be evaluated by elementary means: either make the standard substitution ($x = 2\sin\theta$) or simply note that the integral represents the area of a semi-circle of radius 2.

Problem Sheet 3

1. Evaluate the iterated integrals

   (a) $\int_0^1 dx \int_0^2 dy(x + 3)$

   (b) $\int_0^{\log 3} dx \int_0^{\log 2} dye^{x+y}$

   (c) $\int_0^{\log 2} dx \int_0^1 dyxye^{y^2x}$

   (d) $\int_0^{\pi} d\theta \int_0^{1-\sin\theta} drr^2\cos\theta$

   *Solution:*The first three are very straight-forward, you treat x as a constant when doing the dy integral and visa versa:
$$\int_0^1 dx \int_0^2 dy(x + 3) = \int_0^1 dx(x + 3)y]_0^2 = 2\int_0^1 dx(x + 3) = x^2 + 6x\big]_0^1 = 7 \tag{30}$$

and

$$\int_0^{\log 3} dx \int_0^{\log 2} dy e^{x+y} = \int_0^{\log 3} dx\ e^{x+y}\Big]_0^{\log 2} = 2\int_0^{\log 3} dx e^x = 4 \tag{31}$$

and, for the next one use $u = y^2$,

$$\begin{aligned}
\int_0^{\log 2} dx \int_0^1 dy x y e^{y^2 x} &= \frac{1}{2}\int_0^{\log 2} dx \int_0^1 du x e^{ux} && (32)\\
&= \frac{1}{2}\int_0^{\log 2} dx\ e^{ux}]_0^1 = \frac{1}{2}\int_0^{\log 2} dx (e^x - 1) && (33)\\
&= \frac{1}{2}(e^x - x)_0^{\log 2} = \frac{1}{2}(2 - \log 2 - 1) = \frac{1 - \log 2}{2} && (34)
\end{aligned}$$

The last one is different in that it doesn't have constant limits, but, again, you just do the integrals one by one:

$$\int_0^\pi d\theta \int_0^{1-\sin\theta} dr r^2 \cos\theta = \int_0^\pi d\theta \cos\theta \frac{r^3}{3}\Big]_0^{1-\sin\theta} = \frac{1}{3}\int_0^\pi d\theta \cos\theta (1-\sin\theta)^3 = 0 \tag{35}$$

where we know the integral is zero because the integrand is odd about $\theta = \pi/2$.

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$\begin{aligned}
x &= a\cosh u \cos v && (36)\\
y &= a \sinh u \sin v. && (37)
\end{aligned}$$

Solution:$\delta A = J\delta u \delta v$ with

$$\begin{aligned}
J &= \left|\left|\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{array}\right|\right| = \left|\left|\begin{array}{cc} a\sinh u \cos v & -a\cosh u \sin v \\ a\cosh u \sin v & a \sinh u \cos v\end{array}\right|\right| \\
&= a^2(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v) && (38)
\end{aligned}$$

This can be simplified a bit:

$$\begin{aligned}
J &= a^2(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v) = a^2[\sinh^2 u(1 - \sin^2 v) + \cosh^2 u \sin^2 v] \\
&= a^2[\sinh^2 + \sin^2 v(\cosh^2 - \sinh^2 u)] && (39)
\end{aligned}$$

Using $\cosh^2 u - \sinh^2 u = 1$ gives $J = a^2(\sinh^2 u + \sin^2 v)$.

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos\theta$ (r and θ are polar coordinates).

*Solution:*Use polar coördinates to evaluate area integral; θ ranges from 0 to 2π and r ranges from 0 to $1 + \cos\theta$ and the Jacobian is $J = r$

$$A = \int_D dV = \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r$$

$$\begin{aligned}
&= \int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^2 \\
&= \frac{1}{2}\int_0^{2\pi} d\theta\ (1 + 2\cos\theta + \cos^2\theta) \\
&= \frac{1}{2}(2\pi + 0 + \pi) = \frac{3}{2}\pi, && (40)
\end{aligned}$$

since $\cos\theta$ integrates to zero and the average value of $\cos^2\theta$ is $\frac{1}{2}$.

Similarily

$$\begin{aligned}
\int_D x dV &= \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r^2 \cos\theta \\
&= \frac{1}{3}\int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^3\cos\theta \\
&= \frac{1}{3}\int_0^{2\pi} d\theta\ (\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta) \\
&= \frac{1}{3}\left(3\pi + 0 + \frac{3}{4}\pi\right) = \frac{5}{4}\pi, && (41)
\end{aligned}$$

and so $\bar{x} = 5/6$. By symmetry $\bar{y} = 0$.

4. Evaluate the double integral

$$\int\int_R dA x(1+y^2)^{-1/2} \tag{42}$$

where R is the region in with $x \geq 0$ and $y \geq 0$ enclosed by $y = x^2$, $y = 4$ and $x = 0$.

*Solution:*So the first thing is to iterate the integral and put in the limits. For given y x goes from 0 to $\sqrt{y}$ so we get

$$\int\int_R dA x(1+y^2)^{-1/2} = \int_0^4 dy \int_0^{\sqrt{y}} dx \frac{x}{\sqrt{1+y^2}} = \frac{1}{2}\int_0^4 dy \frac{y}{\sqrt{1+y^2}} \tag{43}$$

and then use $u = 1 + y^2$ to get

$$\int\int_R dA x(1+y^2)^{-1/2} = \frac{1}{4}\int_1^{17} dy \frac{1}{\sqrt{u}} = \frac{1}{2}\sqrt{u}\Big]_1^{17} = \frac{\sqrt{17}-1}{2} \tag{44}$$

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{45}$$

has zero divergence. Here, as usual, $\mathbf{r}$ is the position vector $\mathbf{r} = (x, y, z)$ and $\hat{\mathbf{r}}$ is the corresponding unit vector $\hat{\mathbf{r}} = (x/r, y/r, z/r)$. $r = \sqrt{x^2 + y^2 + z^2}$, again, as usual.

*Solution:*So

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} \tag{46}$$

Using the product rule

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5} \tag{47}$$

and so

$$\nabla \cdot \mathbf{F} = \frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} = 0 \tag{48}$$

using $r^2 = x^2 + y^2 + z^2$. Note by the way we have used

$$\frac{\partial}{\partial x} r = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \tag{49}$$

using the chain rule.