231 Outline Solutions Tutorial Sheet 10, 11 and 12.[1,2]

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Problem Sheet 10

1. Find the Fourier series representation of the sawtooth function f defined by $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

   Solution: f is odd so $a_n = 0$ for all n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x\ \sin nx = -\left.\frac{x\cos nx}{n\pi}\right|_{\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nx}{n}.$$

   The integral on the RHS is zero since it is just a cosine integrated over a full period (or n periods). Thus $b_n = -2\cos(n\pi)/n = -2(-1)^n/n$ which gives

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

2. Establish that

$$\int_{-\pi}^{\pi} dx\ \sin mx\ \sin nx = \int_{-\pi}^{\pi} dx\ \cos mx\ \cos nx = 0,$$

   if $m \neq n$ (both m and n are integers).

   Solution: In this question m and n will be taken as positive integers. The problem can be tackled using complex exponentials or trig identities. Using the identity

$$2\sin A\sin B = \cos(A-B) - \cos(A+B),$$

$$\int_{-\pi}^{\pi} dx\ \sin mx\sin nx = \frac{1}{2}\int_{-\pi}^{\pi} dx\ \left[\cos(m-n)x - \cos(m+n)x\right],$$

   which is zero (integral of cosine over full periods) provided $m - n$ and $m + n$ are non-zero. To show that

$$\int_{-\pi}^{\pi} dx\ \cos mx\cos nx = 0,$$

   use

$$2\cos A\cos B = \cos(A+B) + \cos(A-B).$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2] Including material from Chris Ford, to whom many thanks.

3. The periodic function f is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

and $f(x + 2\pi) = f(x)$.

(a) Represent $f(x)$ as a Fourier series.

Solution: This function is neither odd nor even, though the only non-zero b_n coefficient is $b_1 = \frac{1}{2}$ (since $f(x) = \frac{1}{2}\sin x + |\sin x|$ and $|\sin x|$ is even). Now to the a_n coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \cos nx\ f(x) = \frac{1}{\pi}\int_{0}^{\pi} dx\ \cos nx\ \sin x$$

This can be computed via complex exponentials or through the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$:

$$a_n = \frac{1}{2\pi}\int_0^{\pi} dx\ [\sin(1+n)x + \sin(1-n)x] = -\frac{1}{2\pi}\left(\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right)\Bigg|_0^{\pi}.$$

Now $\cos(1+n)\pi = \cos(1-n)\pi = -(-1)^n$, and so

$$a_n = -\frac{1}{2\pi}(-(-1)^n - 1)\left(\frac{1}{1+n} + \frac{1}{1+n}\right) = \frac{1}{\pi}(1 + (-1)^n)\frac{1}{1-n^2}.$$

This is ambiguous for $n = 1$; it is trivial to check that $a_1 = 0$. Putting everything together

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n>0,\text{even}} \frac{\cos nx}{1-n^2} + \frac{1}{2}\sin x,$$

or

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{1-4m^2} + \frac{1}{2}\sin x.$$

(b) Derive the remarkable formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots = \frac{1}{2}.$$

Solution: $f(0) = 0$ leads to the amazing formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots = \frac{1}{2}.$$

Problem Sheet 11

1. Express the following periodic functions ($l = 2\pi$) as complex Fourier series

   (a)

   $$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

   where $a \in (0, \pi)$ is a constant.
   Solution: $f(x) = \sum_{n \in Z} c_n e^{inx}$ with

   $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\ e^{-inx} f(x) = \frac{1}{2\pi} \int_{-a}^{a} dx\ e^{-inx}$$

   so that $c_0 = a/\pi$ and

   $$c_n = \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_{-a}^{a} = \frac{1}{\pi n} \frac{e^{ian} - e^{-ian}}{2i} = \frac{1}{\pi n} \sin an.$$

   (b)

   $$f(x) = \frac{1}{2 - e^{ix}}.$$

   Solution: This can be expanded as a geometric series which is exactly the complex Fourier series!

   $$f(x) = \frac{1}{2 - e^{ix}} = \frac{1}{2} \frac{1}{1 - \frac{1}{2} e^{ix}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} e^{inx}.$$

2. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ has the Fourier series expansion

   $$f(x) = -\frac{4}{\pi} \sum_{n>0,\ \text{odd}} \frac{\cos nx}{n^2}.$$

   Solution: f is even so $b_n = 0$ for all n.

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\ \cos nx \left(|x| - \frac{1}{2}\pi \right).$$

   A quick calculation gives $a_0 = 0$. For $n > 0$ use the fact that $\cos nx$ integrates to zero over a full period

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\ |x|\ \cos nx = \frac{2}{\pi} \int_{0}^{\pi} dx\ x \cos nx$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left(\frac{x \sin nx}{n}\Big|_0^\pi - \int_0^\pi dx\, \frac{\sin nx}{n} \right) \\
&= 0 + \frac{2}{\pi}\frac{\cos nx}{n^2}\Big|_0^\pi = \frac{2}{\pi}\frac{((-1)^n - 1)}{n^2}.
\end{aligned}
$$

Thus $a_n = 0$ if n is even and $a_n = -4/(\pi n^2)$ if n odd.

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + ...$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

*Solution:*To compute S_1 set $x = \pi/4$ in the Fourier series quoted in question 1

$$f\left(\frac{\pi}{4}\right) = -\frac{4}{\pi}\frac{1}{\sqrt{2}}S_1.$$

Since $f(\frac{\pi}{4}) = -\frac{\pi}{4}$ one has

$$-\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\frac{4}{\pi}S_1,$$

so that

$$S_1 = \frac{\sqrt{2}\pi^2}{16}.$$

ii) The average value of $|f|^2$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, |f(x)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, \left(|x| - \frac{1}{2}\pi\right)^2 = \frac{1}{\pi}\int_0^{\pi} dx\, \left(x - \frac{1}{2}\pi\right)^2.$$

A short calculation gives that this is equal to $\pi^2/12$. Applying Parseval's theorem

$$\frac{\pi^2}{12} = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right) = \frac{1}{2}\cdot\frac{16}{\pi^2}S_2,$$

and so

$$S_2 = \frac{\pi^4}{96}.$$

4. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \, \frac{\cos p}{1+p^2} = \frac{\pi}{e}.$$

Solution:

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{-ikx} \; f(x) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{-ikx} \; e^{-a|x|} = \frac{1}{2\pi}\left[\int_{0}^{\infty} dx \; e^{-ikx-ax} + \int_{-\infty}^{0} dx \; e^{-ikx+ax}\right] \\
&= \frac{1}{2\pi}\left[-\frac{e^{-x(a+ik)}}{a+ik}\bigg|_{0}^{\infty} - \frac{e^{x(a-ik)}}{a-ik}\bigg|_{-\infty}^{0}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \frac{1}{\pi}\frac{a}{a^2+k^2}.
\end{aligned}$$

f can be represented as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk \; e^{ikx} \; \tilde{f}(k) = \frac{a}{\pi}\int_{-\infty}^{\infty} dk \; \frac{e^{ikx}}{a^2+k^2}.$$

Setting $a = 1$ and $x = 1$ gives

$$e^{-1} = \frac{1}{\pi}\int_{-\infty}^{\infty} dk \; \frac{e^{ik}}{1+k^2}.$$

Taking the real part (and multiplying by π)

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} dk \; \frac{\cos k}{1+k^2}.$$

Problem Sheet 12

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1).$$

(a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x) = x$ for $-\pi < x < \pi$ compute

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

*Solution:*From earlier calculations, the Fourier coeficients for the sawtooth are $a_n = 0$ and $b_n = -2(-1)^n/n$. Applying Parsevals theorem:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx\ x^2 = \frac{1}{2}\sum_{n=1} \frac{4}{n^2} = 2\zeta(2)$$

The LHS is just $\pi^2/3$ which gives $\zeta(2) = \pi^2/6$.

(b) Consider the Fourier expansion of $f(x) = x^2, \quad -\pi < x < \pi$, and use the result to show that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

*Solution:*Consider $f(x) = x^2 \qquad -\pi < x < \pi$ an even function so that $b_n = 0$. The a_n can be obtained in the usual way (although one must integrate by parts twice). An alternative way is to integrate the Fourier series for the sawtooth

$$x = -2\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} \qquad -\pi < x < \pi$$

Integration with respect to x yields

$$\frac{x^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} + C,$$

where C is a constant of integration. This constant can be determined by integrating both sides from $x = -\pi$ to $x = \pi$:

$$\left.\frac{x^3}{6}\right|_{-\pi}^{\pi} = 2\pi C,$$

which gives $C = \pi^2/6$. According to Parseval's theorem the average value of $|f(x)|^2$ is given by the sum

$$\frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right),$$

where a_n and b_n are the Fourier coefficients of f. For $f(x) = x^2$, $a_n = 4(-1)^n/n^2$ for $n > 0$ and $a_0 = 4C = 2\pi^2/3$. The average value of $|f(x)|^2 = x^4$ is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\ dx = \frac{\pi^4}{5}.$$

Applying Parseval's theorem

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\zeta(4),$$

and so

$$\zeta(4) = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{90}.$$

2. Determine the Fourier transform of the Gaussian function

$$f(x) = e^{-\alpha x^2},$$

where α is a positive constant.

*Solution:*Completing the square

$$\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2 + \beta x} = \int_{-\infty}^{\infty} dx\ e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \frac{1}{4}\beta^2/\alpha}.$$

Making the change of variables $y = x - \frac{\beta}{2\alpha}$ gives

$$\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2 + \beta x} = e^{\frac{1}{4}\beta^2/\alpha}\int_{-\infty}^{\infty} dy\ e^{-\alpha y^2} = e^{\frac{1}{4}\beta^2/\alpha}\sqrt{\frac{\pi}{\alpha}},$$

using the standard Gaussian integral formula. This derivation assumes that β is real. However, we assume the result is valid for complex β to compute the Fourier transform of $f(x) = e^{-\alpha x^2}$:

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-ikx}e^{-\alpha x^2} = \frac{1}{2\pi}\cdot\sqrt{\frac{\pi}{\alpha}}e^{-\frac{1}{4}k^2/\alpha},$$

by formally taking $\beta = -ik$.

3. Compute

(a)

$$\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3)$$

(b)

$$\int_{-\infty}^{\infty} dx\ \delta(x^2 + x)$$

(c)

$$\int_0^2 dx\ e^x\ \delta'(x-1)$$

(d)

$$\int_0^\infty dx\ e^{-ax}\delta(\cos x)$$

(e)

$$\int_0^\infty dx\ \delta(e^{ax}\cos x).$$

(f)

$$\frac{d}{dx}e^{a\theta(x)}.$$

where a is a constant.

Solution:(a) $\int_{-\infty}^{\infty}\ dx\ x^2\delta(x-3) = 3^2 = 9.$
(b)Use

$$\delta\left(h(x)\right) = \sum_i \frac{\delta(x-x_i)}{|h'(x_i)|},$$

where the x_i are roots of h. Here $h(x) = x^2 + x = x(x+1)$ with roots $x_1 = 0$ and $x_2 = -1$. $h'(x) = 2x+1$ and so $h'(0) = 1$, $h'(-1) = -1$. This gives $\delta(x^2+x) = \delta(x) + \delta(x+1)$

$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x) = 2.$$

(c) Integrate by parts:

$$\int_0^2 dx\ e^x\ \delta'(x-1) = e^x\delta(x-1)|_0^2 - \int_0^2 dx\ e^x\ \delta(x-1) = -e.$$

(d) $h(x) = \cos x$ has zeros at $x = \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi$ etc. and the derivative of $\cos x$ is equal to 1 or -1 at these points. Therefore

$$\int_0^\infty dx\ e^{-ax}\delta(\cos x) = \sum_{n=0}^{\infty} e^{-a(\frac{1}{2}\pi+n\pi)} = e^{-\frac{1}{2}a\pi}\sum_{n=0}^{\infty} e^{-an\pi} = \frac{e^{-\frac{1}{2}a\pi}}{(1-e^{-a\pi})},$$

the last step used the standard geometric series formula. The result may be rewritten in terms of the hyperbolic sine.

$$\int_0^\infty dx\ e^{-ax}\delta(\cos x) = \frac{1}{2\sinh\frac{1}{2}a\pi}.$$

(e) $h(x) = e^{ax}\cos x$, $h'(x) = ae^{ax}\cos x - e^{ax}\sin x$. The zeros of h are the same as in the previous problem. At a zero $|h'(x)| = e^{ax}$. This implies that the integral leads to the same geometric sum as in part (d). (f) First you need to reexpress everything so that it is linear in $\theta(x)$, we can't differenciate powers of $\theta(x)$. So

$$\begin{aligned}\exp a\theta &= \sum_{n=0}^{\infty} \theta(x)^n a^n n! \\ &= 1 + \sum_{n=1}^{\infty} \theta(x)^n a^n n!\end{aligned}$$

then, using $\theta^n = \theta$, easy to check from the definition of θ, we get

$$\begin{aligned}\exp a\theta &= 1 + \theta \sum_{n=1}^{\infty} \frac{a^n}{n!} \\ &= 1 + \theta \left(\sum_{n=0}^{\infty} \frac{\theta(x)^n a^n}{n!} - 1 \right) \\ &= 1 + \theta \left(e^a - 1 \right)\end{aligned}$$

and hence

$$\frac{d}{dx} e^{a\theta(x)} = \frac{d}{dx} \left[1 + \theta \left(e^a - 1 \right) \right] = \delta(x) \left(e^a - 1 \right)$$