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## Problem Sheet 10

1. Find the Fourier series representation of the sawtooth function  $f$  defined by  $f(x) = x$  for  $-\pi < x < \pi$  and  $f(x + 2\pi) = f(x)$ .

*Solution:*  $f$  is odd so  $a_n = 0$  for all  $n$ .

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, x \sin nx = -\frac{x \cos nx}{n\pi} \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos nx}{n}.$$

The integral on the RHS is zero since it is just a cosine integrated over a full period (or  $n$  periods). Thus  $b_n = -2 \cos(n\pi)/n = -2(-1)^n/n$  which gives

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

2. Establish that

$$\int_{-\pi}^{\pi} dx \, \sin mx \sin nx = \int_{-\pi}^{\pi} dx \, \cos mx \cos nx = 0,$$

if  $m \neq n$  (both  $m$  and  $n$  are integers).

*Solution:* In this question  $m$  and  $n$  will be taken as positive integers. The problem can be tackled using complex exponentials or trig identities. Using the identity

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B),$$

$$\int_{-\pi}^{\pi} dx \, \sin mx \sin nx = \frac{1}{2} \int_{-\pi}^{\pi} dx \, [\cos(m - n)x - \cos(m + n)x],$$

which is zero (integral of cosine over full periods) provided  $m - n$  and  $m + n$  are non-zero. To show that

$$\int_{-\pi}^{\pi} dx \, \cos mx \cos nx = 0,$$

use

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$$

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<sup>2</sup>Including material from Chris Ford, to whom many thanks.

3. The periodic function  $f$  is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

and  $f(x + 2\pi) = f(x)$ .

- (a) Represent  $f(x)$  as a Fourier series.

*Solution:* This function is neither odd nor even, though the only non-zero  $b_n$  coefficient is  $b_1 = \frac{1}{2}$  (since  $f(x) = \frac{1}{2} \sin x + |\sin x|$  and  $|\sin x|$  is even). Now to the  $a_n$  coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, \cos nx \, f(x) = \frac{1}{\pi} \int_0^{\pi} dx \, \cos nx \sin x$$

This can be computed via complex exponentials or through the identity

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B):$$

$$a_n = \frac{1}{2\pi} \int_0^{\pi} dx \, [\sin(1 + n)x + \sin(1 - n)x] = -\frac{1}{2\pi} \left( \frac{\cos(1 + n)x}{1 + n} + \frac{\cos(1 - n)x}{1 - n} \right) \Big|_0^{\pi}.$$

Now  $\cos(1 + n)\pi = \cos(1 - n)\pi = -(-1)^n$ , and so

$$a_n = -\frac{1}{2\pi} (-(-1)^n - 1) \left( \frac{1}{1 + n} + \frac{1}{1 - n} \right) = \frac{1}{\pi} (1 + (-1)^n) \frac{1}{1 - n^2}.$$

This is ambiguous for  $n = 1$ ; it is trivial to check that  $a_1 = 0$ . Putting everything together

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n>0, \text{even}} \frac{\cos nx}{1 - n^2} + \frac{1}{2} \sin x,$$

or

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{1 - 4m^2} + \frac{1}{2} \sin x.$$

- (b) Derive the remarkable formula

$$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \dots = \frac{1}{2}.$$

*Solution:*  $f(0) = 0$  leads to the amazing formula

$$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \dots = \frac{1}{2}.$$

## Problem Sheet 11

1. Express the following periodic functions ( $l = 2\pi$ ) as complex Fourier series

(a)

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

where  $a \in (0, \pi)$  is a constant.

*Solution:*  $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$  with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{-inx} f(x) = \frac{1}{2\pi} \int_{-a}^a dx e^{-inx}$$

so that  $c_0 = a/\pi$  and

$$c_n = \frac{1}{2\pi} \frac{e^{-inx}}{-in} \Big|_{-a}^a = \frac{1}{\pi n} \frac{e^{ian} - e^{-ian}}{2i} = \frac{1}{\pi n} \sin an.$$

(b)

$$f(x) = \frac{1}{2 - e^{ix}}.$$

*Solution:* This can be expanded as a geometric series which is exactly the complex Fourier series!

$$f(x) = \frac{1}{2 - e^{ix}} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}e^{ix}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} e^{inx}.$$

2. Show that the periodic function  $f$  defined by  $f(x) = |x| - \frac{1}{2}\pi$  for  $-\pi < x < \pi$  and  $f(x + 2\pi) = f(x)$  has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{\cos nx}{n^2}.$$

*Solution:*  $f$  is even so  $b_n = 0$  for all  $n$ .

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \cos nx \left( |x| - \frac{1}{2}\pi \right).$$

A quick calculation gives  $a_0 = 0$ . For  $n > 0$  use the fact that  $\cos nx$  integrates to zero over a full period

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx |x| \cos nx = \frac{2}{\pi} \int_0^{\pi} dx x \cos nx$$

$$\begin{aligned} &= \frac{2}{\pi} \left( \frac{x \sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} dx \frac{\sin nx}{n} \right) \\ &= 0 + \frac{2}{\pi} \frac{\cos nx}{n^2} \Big|_0^{\pi} = \frac{2}{\pi} \frac{((-1)^n - 1)}{n^2}. \end{aligned}$$

Thus  $a_n = 0$  if  $n$  is even and  $a_n = -4/(\pi n^2)$  if  $n$  odd.

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \dots$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

*Solution:* To compute  $S_1$  set  $x = \pi/4$  in the Fourier series quoted in question 1

$$f\left(\frac{\pi}{4}\right) = -\frac{4}{\pi} \frac{1}{\sqrt{2}} S_1.$$

Since  $f(\pi/4) = -\pi/4$  one has

$$-\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \frac{4}{\pi} S_1,$$

so that

$$S_1 = \frac{\sqrt{2}\pi^2}{16}.$$

ii) The average value of  $|f|^2$  is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dx |f(x)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \left( |x| - \frac{1}{2}\pi \right)^2 = \frac{1}{\pi} \int_0^{\pi} dx \left( x - \frac{1}{2}\pi \right)^2.$$

A short calculation gives that this is equal to  $\pi^2/12$ . Applying Parseval's theorem

$$\frac{\pi^2}{12} = \frac{1}{4} |a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) = \frac{1}{2} \cdot \frac{16}{\pi^2} S_2,$$

and so

$$S_2 = \frac{\pi^4}{96}.$$

4. Compute the Fourier transform of  $f(x) = e^{-a|x|}$  where  $a$  is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \frac{\cos p}{1+p^2} = \frac{\pi}{e}.$$

*Solution:*

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} e^{-a|x|} = \frac{1}{2\pi} \left[ \int_0^{\infty} dx e^{-ikx-ax} + \int_{-\infty}^0 dx e^{-ikx+ax} \right] \\ &= \frac{1}{2\pi} \left[ -\frac{e^{-x(a+ik)}}{a+ik} \Big|_0^{\infty} - \frac{e^{x(a-ik)}}{a-ik} \Big|_{-\infty}^0 \right] \\ &= \frac{1}{2\pi} \left[ \frac{1}{a+ik} + \frac{1}{a-ik} \right] = \frac{1}{\pi} \frac{a}{a^2+k^2}. \end{aligned}$$

$f$  can be represented as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk e^{ikx} \tilde{f}(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{a^2+k^2}.$$

Setting  $a = 1$  and  $x = 1$  gives

$$e^{-1} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{e^{ik}}{1+k^2}.$$

Taking the real part (and multiplying by  $\pi$ )

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} dk \frac{\cos k}{1+k^2}.$$

## Problem Sheet 12

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1).$$

- (a) By applying Parseval's theorem for Fourier series to the sawtooth  $f(x) = x$  for  $-\pi < x < \pi$  compute

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

*Solution:* From earlier calculations, the Fourier coefficients for the sawtooth are  $a_n = 0$  and  $b_n = -2(-1)^n/n$ . Applying Parseval's theorem:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dx x^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2} = 2\zeta(2)$$

The LHS is just  $\pi^2/3$  which gives  $\zeta(2) = \pi^2/6$ .

- (b) Consider the Fourier expansion of  $f(x) = x^2$ ,  $-\pi < x < \pi$ , and use the result to show that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

*Solution:* Consider  $f(x) = x^2$   $-\pi < x < \pi$  an even function so that  $b_n = 0$ . The  $a_n$  can be obtained in the usual way (although one must integrate by parts twice). An alternative way is to integrate the Fourier series for the sawtooth

$$x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} \quad -\pi < x < \pi$$

Integration with respect to  $x$  yields

$$\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} + C,$$

where  $C$  is a constant of integration. This constant can be determined by integrating both sides from  $x = -\pi$  to  $x = \pi$ :

$$\frac{x^3}{6} \Big|_{-\pi}^{\pi} = 2\pi C,$$

which gives  $C = \pi^2/6$ . According to Parseval's theorem the average value of  $|f(x)|^2$  is given by the sum

$$\frac{1}{4} |a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2),$$

where  $a_n$  and  $b_n$  are the Fourier coefficients of  $f$ . For  $f(x) = x^2$ ,  $a_n = 4(-1)^n/n^2$  for  $n > 0$  and  $a_0 = 4C = 2\pi^2/3$ . The average value of  $|f(x)|^2 = x^4$  is given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{5}.$$

Applying Parseval's theorem

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\zeta(4),$$

and so

$$\zeta(4) = \frac{\pi^4}{8} \left( \frac{1}{5} - \frac{1}{9} \right) = \frac{\pi^4}{90}.$$

2. Determine the Fourier transform of the Gaussian function

$$f(x) = e^{-\alpha x^2},$$

where  $\alpha$  is a positive constant.

*Solution:* Completing the square

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = \int_{-\infty}^{\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \frac{1}{4}\beta^2/\alpha}.$$

Making the change of variables  $y = x - \frac{\beta}{2\alpha}$  gives

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = e^{\frac{1}{4}\beta^2/\alpha} \int_{-\infty}^{\infty} dy e^{-\alpha y^2} = e^{\frac{1}{4}\beta^2/\alpha} \sqrt{\frac{\pi}{\alpha}},$$

using the standard Gaussian integral formula. This derivation assumes that  $\beta$  is real. However, we assume the result is valid for complex  $\beta$  to compute the Fourier transform of  $f(x) = e^{-\alpha x^2}$ :

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} e^{-\alpha x^2} = \frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\alpha}} e^{-\frac{1}{4}k^2/\alpha},$$

by formally taking  $\beta = -ik$ .

3. Compute

(a)

$$\int_{-\infty}^{\infty} dx x^2 \delta(x-3)$$

(b)

$$\int_{-\infty}^{\infty} dx \delta(x^2 + x)$$

(c)

$$\int_0^2 dx e^x \delta'(x-1)$$

(d)

$$\int_0^{\infty} dx e^{-ax} \delta(\cos x)$$

(e)

$$\int_0^{\infty} dx \delta(e^{ax} \cos x).$$

(f)

$$\frac{d}{dx} e^{a\theta(x)}.$$

where  $a$  is a constant.

*Solution:* (a)  $\int_{-\infty}^{\infty} dx x^2 \delta(x-3) = 3^2 = 9$ .

(b) Use

$$\delta(h(x)) = \sum_i \frac{\delta(x - x_i)}{|h'(x_i)|},$$

where the  $x_i$  are roots of  $h$ . Here  $h(x) = x^2 + x = x(x+1)$  with roots  $x_1 = 0$  and  $x_2 = -1$ .  $h'(x) = 2x + 1$  and so  $h'(0) = 1$ ,  $h'(-1) = -1$ . This gives  $\delta(x^2 + x) = \delta(x) + \delta(x+1)$

$$\int_{-\infty}^{\infty} dx \delta(x^2 + x) = 2.$$

(c) Integrate by parts:

$$\int_0^2 dx e^x \delta'(x-1) = e^x \delta(x-1) \Big|_0^2 - \int_0^2 dx e^x \delta(x-1) = -e.$$

(d)  $h(x) = \cos x$  has zeros at  $x = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi$  etc. and the derivative of  $\cos x$  is equal to 1 or -1 at these points. Therefore

$$\int_0^{\infty} dx e^{-ax} \delta(\cos x) = \sum_{n=0}^{\infty} e^{-a(\frac{1}{2}\pi + n\pi)} = e^{-\frac{1}{2}a\pi} \sum_{n=0}^{\infty} e^{-an\pi} = \frac{e^{-\frac{1}{2}a\pi}}{(1 - e^{-a\pi})},$$

the last step used the standard geometric series formula. The result may be rewritten in terms of the hyperbolic sine.

$$\int_0^{\infty} dx e^{-ax} \delta(\cos x) = \frac{1}{2 \sinh \frac{1}{2}a\pi}.$$

(e)  $h(x) = e^{ax} \cos x$ ,  $h'(x) = ae^{ax} \cos x - e^{ax} \sin x$ . The zeros of  $h$  are the same as in

the previous problem. At a zero  $|h'(x)| = e^{ax}$ . This implies that the integral leads to the same geometric sum as in part (d). (f) First you need to reexpress everything so that it is linear in  $\theta(x)$ , we can't differentiate powers of  $\theta(x)$ . So

$$\begin{aligned}\exp a\theta &= \sum_{n=0}^{\infty} \theta(x)^n a^n n! \\ &= 1 + \sum_{n=1}^{\infty} \theta(x)^n a^n n!\end{aligned}$$

then, using  $\theta^n = \theta$ , easy to check from the definition of  $\theta$ , we get

$$\begin{aligned}\exp a\theta &= 1 + \theta \sum_{n=1}^{\infty} \frac{a^n}{n!} \\ &= 1 + \theta \left( \sum_{n=0}^{\infty} \frac{\theta(x)^n a^n}{n!} - 1 \right) \\ &= 1 + \theta (e^a - 1)\end{aligned}$$

and hence

$$\frac{d}{dx} e^{a\theta(x)} = \frac{d}{dx} [1 + \theta (e^a - 1)] = \delta(x) (e^a - 1)$$