231 Tutorial Sheet 9 due: Thursday 17 January[1,2]

10 January 2007

Useful facts:

- Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{N}\times\mathbf{dl}$ points into S with $\mathbf{n}$ the normal and $\mathbf{dl}$ a tangent to C and $\mathbf{F}$ a vector field defined in a region containing S, then

$$\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{dA} = \int_C \mathbf{F}\cdot\mathbf{dl} \tag{1}$$

  item $\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{dA}$ is called the *flux* of $\mathbf{F}$ across S.

- If S is a surface δS is its bounding curve with the orientation describe above.

- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if $f(x,y)$ and $g(x,y)$ have continuous first derivatives

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_C (fdx + gdy) \tag{2}$$

- The Gauss Divergence Theorem: If D is connected region in $\mathbf{R}^3$ with a piecewise smooth surface S oriented to point out of D and if $\mathbf{F}$ is a vector field defined in a region containing D and with continuous derivatives then

$$\int_D dV\nabla\cdot\mathbf{F} = \int_S \mathbf{F}\cdot\mathbf{dA} \tag{3}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Let S be the closed surface consisting of the portion of the paraboloid $z = x^2 + y^2$ for which $0 \leq z \leq 1$ and capped by the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j} - y\mathbf{k}$ in the outrward direction across S.

2. For any conduction loop C the electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$ are related by
$$\oint_c \mathbf{E} \cdot d\mathbf{r} = -\int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{4}$$
where $C = \delta S$, use Stoke's theorem to find a differential equation relating $\mathbf{E}$ and $\mathbf{B}$.

3. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

4. Consider, again, the vector field
$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$
   (a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.
   (b) Compute the flux of $\mathbf{F}$ out of the box $1 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
   (c) Compute the flux of $\mathbf{F}$ out of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

5. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$