231 Tutorial Sheet 8[1][2]

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Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- To evaluate the surface integral for a parameterized surface:

$$\int\int_S \mathbf{F}\cdot\mathbf{dA} = \int\int_S \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}\right)dudv \tag{2}$$

- Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{N}\times\mathbf{dl}$ points into S with $\mathbf{n}$ the normal and $\mathbf{dl}$ a tangent to C and $\mathbf{F}$ a vector field defined in a region containing S, then

$$\int_S \mathrm{curl}\,\mathbf{F}\cdot\mathbf{dA} = \int_C \mathbf{F}\cdot\mathbf{dl} \tag{3}$$

- If S is a surface δS is its bounding curve with the orientation describe above.

- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if $f(x,y)$ and $g(x,y)$ have continuous first derivatives

$$\int_D \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy = \int_C (fdx+gdy) \tag{4}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the portion of the paraboloid
$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (1 - u^2)\mathbf{k} \tag{5}$$
with $1 \leq u \leq 2$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

2. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$ across the portion of the paraboloid
$$\mathbf{r}(u, v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{6}$$
with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

3. Use Green's Theorem to evaluate
$$\oint_c (y^2 dx + x^2 dy) \tag{7}$$
where C is the square with vertice $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ and oriented anti-clockwise.

4. Calculate directly and using Stoke's Theorem
$$\int_S \mathbf{F} \cdot \mathbf{dS} \tag{8}$$
where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.