231 Tutorial Sheet 8[1][2]

6 December 2007

Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F} \cdot \mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- To evaluate the surface integral for a parameterized surface:

$$\int\int_S \mathbf{F} \cdot \mathbf{dA} = \int\int_S \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} \right) dudv \tag{2}$$

- Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{N} \times \mathbf{dl}$ points into S with $\mathbf{n}$ the normal and $\mathbf{dl}$ a tangent to C and $\mathbf{F}$ a vector field defined in a region containing S, then

$$\int_S \text{curl}\, \mathbf{F} \cdot \mathbf{dA} = \int_C \mathbf{F} \cdot \mathbf{dl} \tag{3}$$

- If S is a surface δS is its bounding curve with the orientation describe above.

- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if $f(x, y)$ and $g(x, y)$ have continuous first derivatives

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_C (fdx + gdy) \tag{4}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + (1 - u^2)\mathbf{k} \tag{5}$$

with $1 \leq u \leq 2$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

2. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x \sin z\mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u, v) = 2 \cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{6}$$

with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

3. Use Green's Theorem to evaluate

$$\oint_c (y^2 dx + x^2 dy) \tag{7}$$

where C is the square with vertice $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ and oriented anti-clockwise.

4. Calculate directly and using Stoke's Theorem

$$\int_S \mathbf{F} \cdot \mathbf{dS} \tag{8}$$

where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.