231 Tutorial Sheet 7[1][2]

22 November 2007

Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- A vector field $\mathbf{F}$ is **conservative** if $\mathbf{F} = \text{grad}\,\phi$ for some scalar field ϕ. ϕ is often called a **potential** for $\mathbf{F}$.

- A vector field is path-independent if its line integral between any two points is independent of the path.

- An irrotational field on a simply connected domain is conservative.

- To find a potential by integration: let $\partial_x\phi = F_1$ and integrate to find ϕ determined up to an arbitrary function of y and z, substitute back into $\partial_y\phi = F_2$ to determine it up to an arbitary function of z and then determine this, up to an arbitrary constant, by substituting into $\partial_z\phi = F_3$

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{2}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- To evaluate the surface integral for a parameterized surface:

$$\int\int_S \mathbf{F}\cdot\mathbf{dA} = \int\int_S \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}\right)dudv \tag{3}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Which of the following vector fields are conservative?
   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.
   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.
   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ where $\mathbf{B}$ is a constant vector.

2. Find a potential for

$$\mathbf{F} = (e^{yz} + zye^{xy})\mathbf{i} + (xze^{yz} + xze^{xy})\mathbf{j} + (xye^{yz} + e^{xy})\mathbf{k} \tag{4}$$

3. Compute the flux of the vector field $\mathbf{F} = (x + x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \leq z \leq 1$.

4. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the $z = 0$ plane.