231 Tutorial Sheet 6: due Thursday November 22[1,2]

15 November 2007

Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- A vector field $\mathbf{F}$ is **conservative** if $\mathbf{F}$ =grad ϕ for some scalar field ϕ. ϕ is often called a **potential** for $\mathbf{F}$.

- A vector field is path-independent if its line integral between any two points is independent of the path.

- An irrotational field on a simply connected domain is conservative.

- To find a potential by integration: let $\partial_x\phi = F_1$ and integrate to find ϕ determined up to an arbitrary function of y and z, substitute back into $\partial_y\phi = F_2$ to determine it up to an arbitary function of z and then determine this, up to an arbitrary constant, by substituting into $\partial_z\phi = F_3$

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. For each of the following vector fields compute the line integral $\int_C \mathbf{F} \cdot \mathbf{dl}$ where C is the semi-circle of radius two around the origin in the xy-plane with y positive, taken anti-clockwise.

   (a) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$

   (b) $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

2. Evaluate the line integrals $\int_C \mathbf{F} \cdot \mathbf{dl}$ for

   (a) $\mathbf{F} = (x^2 y, 4, 0)$ with C given by $\mathbf{r}(t) = (\exp(t), \exp(-t), 0)$ with t going from zero to one;

   (b) $\mathbf{F} = (z, x, y)$ with C given by $\mathbf{r}(t) = (\sin t, 3\sin t, \sin^2 t)$ with $t$going from zero to $\pi/2$.

   (c) $\mathbf{F} = \lambda(x, y)$ with $\lambda = (x^2 + y^2)^{-3/2}$ and with C given by $\mathbf{r}(t) = (e^t \sin t, e^t \cos t)$ with t going from zero to one.

3. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla\phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = 3y^2\mathbf{i} + 6xy\mathbf{j}$

   (c) $\mathbf{F} = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$

   (d) $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j}$

4. Consider the 'point vortex' vector field
$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}.$$
   Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.