231 Tutorial Sheet 4[1,2]

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Useful facts:

- For a scalar field ϕ the gradiant is

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{1}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the divergence is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{2}$$

- The gradient of the divergence of a scalar field is called the Laplacian:

$$\triangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the curl is

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{4}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is
$$J = r^2 \sin\theta. \tag{5}$$
Consider the hemisphere defined by
$$\begin{aligned} \sqrt{x^2+y^2+z^2} &\leq 1 \\ z &\geq 0 \end{aligned} \tag{6}$$
Using spherical polar coordinates compute its volume and centroid.

2. Show $\operatorname{div}\mathbf{r} = 3$ and $\operatorname{grad}|\mathbf{r}| = \mathbf{r}/|\mathbf{r}|$.

3. Find $\nabla(1/|\mathbf{r}|)$.

4. Show $\operatorname{grad} f(r) = f'(r)\hat{\mathbf{r}}$ where $r = |\mathbf{r}|$. If $\mathbf{F}(r) = f(r)\mathbf{r}$ find $\operatorname{div}\mathbf{F}(r)$. Find $\operatorname{div}\operatorname{grad} f(r)$.

5. Show that away from the origin the vector field
$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{7}$$
is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$).