231 Tutorial Sheet 3: due Friday November 2[1][2]

26 October 2007

Useful facts:

- The Jacobian in three-dimensions:

$$dx_1 dx_2 dx_3 = J dy_1 dy_2 dy_3 \tag{1}$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \left\| \begin{array}{ccc} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{array} \right\| \tag{2}$$

- Some trigonometric integrals are required. In particular you may quote the integrals:

$$\begin{aligned} \int_0^{2\pi} d\theta \ \cos\theta &= 0, \\ \int_0^{2\pi} d\theta \ \cos^2\theta &= \pi, \\ \int_0^{2\pi} d\theta \ \cos^3\theta &= 0 \\ \int_0^{2\pi} d\theta \ \cos^4\theta &= \frac{3}{4}\pi \end{aligned} \tag{3}$$

Two are zero by symmetry, the other two can be computed through standard trigonometric identities or via complex exponentials:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{4}$$

- For a scalar field ϕ the gradiant is

$$\mathrm{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{5}$$

- If $\mathbf{a}$ is a vector $\hat{\mathbf{a}}$ is the corresponding unit vector

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} \tag{6}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the divergence is

$$\mathrm{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{7}$$

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[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Evaluate the iterated integrals
   (a) $\int_0^1 dx \int_0^2 dy(x+3)$
   (b) $\int_0^{\log 3} dx \int_0^{\log 2} dy e^{x+y}$
   (c) $\int_0^{\log 2} dx \int_0^1 dy xy e^{y^2 x}$
   (d) $\int_0^{\pi} d\theta \int_0^{1-\sin\theta} dr r^2 \cos\theta$
2. Compute the element of area for elliptic cylinder coordinates which are defined as
$$x = a \cosh u \cos v \quad (8)$$
$$y = a \sinh u \sin v. \quad (9)$$
3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos\theta$ (r and θ are polar coordinates).
4. Evaluate the double integral
$$\int\int_R dA x(1+y^2)^{-1/2} \quad (10)$$
where R is the region in with $x \geq 0$ and $y \geq 0$ enclosed by $y = x^2$, $y = 4$ and $x = 0$.
5. Show that away from the origin the vector field
$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \quad (11)$$
has zero divergence. Here, as usual, $\mathbf{r}$ is the position vector $\mathbf{r} = (x, y, z)$ and $\hat{\mathbf{r}}$ is the corresponding unit vector $\hat{\mathbf{r}} = (x/r, y/r, z/r)$. $r = \sqrt{x^2+y^2+z^2}$, again, as usual.