231 Tutorial Sheet 2[1][2]

18 October 2007

Useful facts:

- The *iterated integral* is the integral expressed as a series of nested one-dimensional integrals.
- The two-dimensional area element $dA = dxdy = rdrd\theta$
- The Jacobian in two-dimensions:

$$dx_1 dx_2 = J dy_1 dy_2 \tag{1}$$

where

$$J = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \left\| \begin{array}{cc} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{array} \right\| \tag{2}$$

Questions

1. Consider the integral

$$I = \int_D dV\ \phi \tag{3}$$

where D is the interior of the ellipsoid defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{4}$$

Write down I as an iterated triple integral.

2. The Gaussian integral formula

$$\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi} \tag{5}$$

can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over R^2:

$$\left(\int_{-\infty}^{\infty} dx\ e^{-x^2} \right)^2 = \int_{R^2} dA\ e^{-x^2-y^2}. \tag{6}$$

By changing to polar coordinates evaluate this integral.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

3. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv.$$

4. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*