231 Tutorial Sheet 18, due Thursday 17 April<sup>12</sup>

## 20 April 2008

Useful facts:

• Series solution: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and, by substituting into the equation find a recursion relation: an equation relating higher terms in  $a_n$  to lower one.

- By expanding out the sum it is easy to see  $y(0) = a_0$  and  $y'(0) = a_1$
- The method of Froebenius: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

under the assumption that  $a_0 \neq 0$ , this assumption will give an equation for s called the indicial equation.

• If y = u solves y'' + py' + qy = 0 then a second solution is given by y = uv where  $v'' + (p + 2\log u')v' = 0$ .

## Questions

1. Bessel's equation reads

$$x^{2}y'' + xy' + (x^{2} - \nu^{2})y = 0.$$

In the lectures it was shown that inserting a Frobenius series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

with  $a_0 \neq 0$  leads to  $a_1 = 0$ , the indical equation,  $s^2 - \nu^2 = 0$ , and the recursion relation  $a_{\nu}$ 

$$a_{n+2} = -\frac{a_n}{(n+s+2)^2 - \nu^2}.$$

For  $\nu = 0$  this leads to two equal roots s = 0 and so the method only provides one solution. Use the recursion relation to compute the  $a_n$  for this case.

2. Use the method of Frobenius to obtain the general solution to the ODE

$$4xy''(x) + 2y'(x) + y(x) = 0.$$

3. In each of the following cases find a second solution in the form y(x) = u(x)v(x) where u(x) is a solution and v(x) is to be determined.

(a) y'' + 6y' + 5y = 0; one solution is  $u(x) = e^{-2x}$ .

(b)  $(1 - x^2)y'' - 2xy' = 0$  one solution is u(x) = 1.

Remark: Part b) is the  $\alpha = 0$  case of Legendre's equation.

4. The Legendre polynomials  $P_n(x)$  are generated by

$$\Phi(x,h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{1}$$

Write down the first four Legendre polynomials and verify that they are orthogonal

$$\int_{-1}^{1} P_n(x) P_m(x) dx = 0$$
(2)

for  $n \neq m$ .

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