231 Tutorial Sheet 15, due Thursday 6 March.[1,2]

28 February 2008

Useful fact:

- To solve the equation $ay''+by'+cy=0$, with a, b and c constants, use an exponential substitution $y=\exp(\lambda x)$ and solve for λ. If this only gives one solution, then $y=x\exp(\lambda x)$ is also a solution.
- The general solution of the equation $ay''+by'+cy=d\exp(\mu x)$, with a, b, c and d constants is of the form $y=y_p+y_c$ where y_c is the general solution to $ay''+by'+cy=0$. To find y_p, substitute $y_p=C\exp\mu x$ and solve for C. This doesn't work is $exp\mu x$ solves $ay''+by'+cy=0$, in which case substitute $y_p=Cx\exp\mu x$, this in turn won't work if $x\exp\mu x$ also solves $ay''+by'+cy=0$, in which case substitute $y_p=Cx^2\exp\mu x$.
- To solve $ay''+by'+cy=f(x)$ where $f(x)$ isn't an exponetial, then write $f(x)$ in terms of exponentials, using Fourier methods if needed. For example, say

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

then find a particular solution y_n to the equation

$$ay''+by'+cy=c_n e^{inx}$$

for all n, the general solution to $ay''+by'+cy=f(x)$ is then

$$y=y_c+\sum_{n=-\infty}^{\infty}y_n$$

where, again, y_c solves the corresponding homogeneous equation.

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Obtain the general solutions of the ODEs
   (a) $y''+3y'-4y=e^{-x}$
   (b) $y''+3y'-4y=e^{-4x}$
   (c) $y''+3y'-4y=\sinh x$
2. Obtain the general solution of the ODE

$$y''(x)+3y'(x)+2y(x)=f(x)$$

where f is the periodic function defined by

$$f(x)=\begin{cases}0 & -\pi<x<-a\\ 1 & -a<x<a\\ 0 & a<x<\pi\end{cases}$$

where $a\in(0,\pi)$ is a constant and $f(x+2\pi)=f(x)$.

3. Obtain the general solutions of the ODEs
   (a) $y''+y=f(x)$, where f is the periodic square wave defined by

$$f(x)=\begin{cases}1, & 0<x<\pi\\ -1, & -\pi<x<0\end{cases}\quad\text{and } f(x+2\pi)=f(x)$$

   (b) $y''+y'+3y=e^{-|x|}$.