231 Tutorial Sheet 13.[1][2]

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Useful facts:

- The Fourier integral or Fourier transform:

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx} \\
\widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}
\end{aligned}
$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x) = f(0)$$

- The Gauss Divergence Theorem: If D is connected region in $\mathbf{R}^3$ with a piecewise smooth surface S oriented to point out of D and if $\mathbf{F}$ is a vector field defined in a region containing D and with continuous derivatives then

$$\int_D dV \nabla \cdot \mathbf{F} = \int_S \mathbf{F} \cdot \mathbf{dA} \tag{1}$$

Questions

1. Express the following functions as Fourier integrals:

   (a)

$$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

   (b)

$$f(x) = \frac{\sin x}{x}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

2. Prove the following properties of the Fourier transform

   (a) The Fourier transform of an even function is even.

   (b) $\tilde{f}'(k) = ik\tilde{f}(k)$.

3. In the lectures (quite a while ago) it was shown that the scalar field

$$\phi(\mathbf{r}) = \frac{1}{r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is harmonic except at the origin. In fact it can be shown that

$$\nabla^2 \phi(\mathbf{r}) = -4\pi\delta^3(\mathbf{r}). \tag{A}$$

where

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

Formally apply Gauss' theorem to the vector field $\mathbf{F} = \nabla\phi$ to show that

$$\int_{r<a} dV\ \nabla^2\ \phi = -4\pi.$$

This is clearly consistent with (A). Another treatment would replace the singular scalar field ϕ with a sequence of smooth scalar fields, e.g.

$$\phi_n(\mathbf{r}) = \frac{n}{\sqrt{n^2 r^2 + 1}}.$$

Prove that

$$\int_{R^3} dV\ \nabla^2 \phi_n(\mathbf{r}) = -4\pi.$$

Suggestion: Use Gauss' theorem to perform the integral for $r < a$ instead of R^3. Then take the $a \to \infty$ limit.