231 Tutorial Sheet 11.[1,2]

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Useful facts:

- A function $f(x)$ has period l if $f(x+l)=f(x)$, it is odd if $f(-x)=-f(x)$ and even if $f(-x)=f(x)$.
- A function with period l has the Fourier series expansion
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi nx}{l}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi nx}{l}\right).$$
where
$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)dx \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2\pi nx}{l}\right)dx \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\sin\left(\frac{2\pi nx}{l}\right)dx \end{aligned}$$
- A function with period l has the Fourier series expansion
$$f(x)=\sum_{n=-\infty}^{\infty} c_n\exp\frac{2\pi nx}{l}.$$
where
$$c_n=\frac{1}{l}\int_{-l/2}^{l/2} f(x)\exp\left(\frac{-2\pi nx}{l}\right)dx$$
- Parceval's formula:
$$\begin{aligned} \frac{1}{l}\int_{-l/2}^{l/2} dx|f(x)|^2 &= \frac{1}{4}a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) \\ &= \sum_{n=-\infty}^{\infty}|c_n|^2 \end{aligned}$$
- The Fourier integral or Fourier transform:
$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\,\widetilde{f(k)}e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)e^{-ikx} \end{aligned}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Express the following periodic functions ($l = 2\pi$) as complex Fourier series

   (a)

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

   where $a \in (0, \pi)$ is a constant.

   (b)

$$f(x) = \frac{1}{2 - e^{ix}}.$$

2. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0,\ \text{odd}} \frac{\cos nx}{n^2}.$$

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + ...$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...$$

   Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

4. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp\, \frac{\cos p}{1 + p^2} = \frac{\pi}{e}.$$