

## 231 Tutorial Sheet 11.<sup>1,2</sup>

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### Useful facts:

- A function  $f(x)$  has period  $l$  if  $f(x+l) = f(x)$ , it is odd if  $f(-x) = -f(x)$  and even if  $f(-x) = f(x)$ .
- A function with period  $l$  has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) dx \\ a_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \cos\left(\frac{2\pi nx}{l}\right) dx \\ b_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \sin\left(\frac{2\pi nx}{l}\right) dx \end{aligned}$$

- A function with period  $l$  has the Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi nx}{l}\right).$$

where

$$c_n = \frac{1}{l} \int_{-l/2}^{l/2} f(x) \exp\left(\frac{-2\pi nx}{l}\right) dx$$

- Parseval's formula:

$$\begin{aligned} \frac{1}{l} \int_{-l/2}^{l/2} dx |f(x)|^2 &= \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned}$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \end{aligned}$$

### Questions

- Express the following periodic functions ( $l = 2\pi$ ) as complex Fourier series

(a)

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

where  $a \in (0, \pi)$  is a constant.

(b)

$$f(x) = \frac{1}{2 - e^{ix}}.$$

- Show that the periodic function  $f$  defined by  $f(x) = |x| - \frac{1}{2}\pi$  for  $-\pi < x < \pi$  and  $f(x+2\pi) = f(x)$  has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{\cos nx}{n^2}.$$

- Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \dots$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

- Compute the Fourier transform of  $f(x) = e^{-a|x|}$  where  $a$  is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \frac{\cos p}{1+p^2} = \frac{\pi}{e}.$$

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<sup>2</sup>Including material from Chris Ford, to whom many thanks.