

231 Tutorial Sheet 10.¹²

17 January 2008

Useful facts:

- A function $f(x)$ has period l if $f(x+l) = f(x)$, it is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$.
- A function with period l has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) dx \\ a_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \cos\left(\frac{2\pi nx}{l}\right) dx \\ b_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \sin\left(\frac{2\pi nx}{l}\right) dx \end{aligned}$$

¹Conor Houghton, houghton@maths.tcd.ie, see also <http://www.maths.tcd.ie/~houghton/231>

²Including material from Chris Ford, to whom many thanks.

Questions

1. Find the Fourier series representation of the sawtooth function f defined by $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.
2. Establish that

$$\int_{-\pi}^{\pi} dx \sin mx \sin nx = \int_{-\pi}^{\pi} dx \cos mx \cos nx = 0,$$

if $m \neq n$ (both m and n are integers).

3. The periodic function f is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

and $f(x + 2\pi) = f(x)$.

- (a) Represent $f(x)$ as a Fourier series.

Remarks: This function is neither odd nor even so both sets of Fourier coefficients are required. However it turns out that all the b_n are zero except for b_1 . Can you see why this is the case without computing an integral?

- (b) Derive the remarkable formula

$$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \dots = \frac{1}{2}.$$

It might be useful to consider $f(0)$.