231 Tutorial Sheet 10.¹²

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Useful facts:

- A function f(x) has period l if f(x+l) = f(x), it is odd if f(-x) = -f(x) and even if f(-x) = f(x).
- A function with period l has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right).$$

where

$$a_0 = \frac{2}{l} \int_{-l/2}^{l/2} f(x) dx$$

$$a_n = \frac{2}{l} \int_{-l/2}^{l/2} f(x) \cos\left(\frac{2\pi nx}{l}\right) dx$$

$$b_n = \frac{2}{l} \int_{-l/2}^{l/2} f(x) \sin\left(\frac{2\pi nx}{l}\right) dx$$

Questions

1. Find the Fourier series representation of the sawtooth function f defined by f(x) = x for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

2. Establish that

$$\int_{-\pi}^{\pi} dx \, \sin mx \, \sin nx = \int_{-\pi}^{\pi} dx \, \cos mx \, \cos nx = 0,$$

if $m \neq n$ (both m and n are integers).

3. The periodic function f is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

- and $f(x + 2\pi) = f(x)$.
- (a) Represent f(x) as a Fourier series.

Remarks: This function is neither odd nor even so both sets of Fourier coefficients are required. However it turns out that all the b_n are zero except for b_1 . Can you see why this is the case without computing an integral?

(b) Derive the remarkable formula

$$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \dots = \frac{1}{2}.$$

It might be useful to consider f(0).

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