231 Outline Solutions Tutorial Sheet 7, 8 and 9.[1,2]

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Problem Sheet 7

1. Which of the following vector fields are conservative?

   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.

   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ where $\mathbf{B}$ is a constant vector.

   Solution:

   (a) $\mathbf{F} = \nabla yz\cos x$ so $\mathbf{F}$ is conservative.

   (b) curl $\mathbf{F} = \mathbf{k} \neq 0$ so $\mathbf{F}$ is not conservative.

   (c) A short calculation gives curl $\mathbf{F} = \mathbf{B}$ so $\mathbf{F}$ is not conservative. Remark: $\frac{1}{2}(\mathbf{B}\times\mathbf{r})$ is a vector potential for the constant vector field $\mathbf{B}$.

2. Compute the flux of the vector field $\mathbf{F} = (x+x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2+y^2=1$ and $0\leq z\leq 1$.

   *Solution:*As we have seen before, $\mathbf{r}(u,v) = \cos u\mathbf{i} + \sin u\mathbf{j} + v\mathbf{k}$ giving

   $$\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = \cos u\mathbf{i} + \sin u\mathbf{j}. \tag{1}$$

   and

   $$\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = (x+x^2)\cos u + y\sin u = \cos^2 + \cos^3 u + \sin^2 u = 1+\cos^3 u. \tag{2}$$

   and so, noting that the $\mathbf{F}$ is perpendicular to the normal at both ends and so we need only include the curved surface

   $$\int_S \mathbf{F}\cdot\mathbf{dA} = \int_0^1 dv\int_0^{2\pi} du\ (1+\cos^3 u) = 2\pi, \tag{3}$$

   since the $\cos^3 u$ integral is zero by symmetry.

   Note: This problem can be solved by noting that $x^2\mathbf{i}$ makes no contribution (by symmetry) and $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has flux 2π since $\mathbf{F}\cdot\mathbf{n} = 1$.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

3. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the $z = 0$ plane.

   *Solution:*To parametrize the sphere choose $(u, v) = (\theta, \phi)$, that is, spherical polar angles. Since the radius $r = a$ this gives $x(u, v) = a \sin u \cos v$, $y(u, v) = a \sin u \sin v$, $z(u, v) = a \cos u$ with $0 \le \theta \le \pi/2$ and $0 \le \phi < 2\pi$. Since $\mathbf{F}$ only has a non-zero z-component we just need

$$\begin{aligned}\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)_3 &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\ &= a^2(\cos u \cos v \sin u \cos v + \cos u \sin v \sin u \sin v) \\ &= a^2 \cos u \sin u, \end{aligned} \tag{4}$$

   which is positive. So the orientation is upwards. Now $F_z = z^3 = a^3 \cos^3 u$, and so

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{A} &= a^5 \int_0^{\frac{1}{2}\pi} du \int_0^{2\pi} dv \ \cos^4 u \sin u \\ &= 2\pi a^5 \int_0^{\frac{1}{2}\pi} du \ \cos^4 u \sin u \\ &= 2\pi a^5 \cdot -\frac{\cos^5 u}{5}\bigg]_0^{\frac{1}{2}\pi} = \frac{2\pi a^5}{5}. \end{aligned} \tag{5}$$

4. Let D be a plane region with area A whose boundary is a piecewise smooth closed curve C. Use Green's theorem to prove that the centroid $(\bar{x}, \bar{y})$ of D is

$$\begin{aligned}\bar{x} &= \frac{1}{2A}\oint_C dy \ x^2 \\ \bar{y} &= -\frac{1}{2A}\oint_C dx \ y^2. \end{aligned} \tag{6}$$

   Use this result to compute the centroid of a semi-circle (this was determined in the lectures using the more standard formula).

   *Solution:*The centroid $(\bar{x}, \bar{y})$ of a plane region D is given by

$$\bar{x} = \frac{\int_D dA \ x}{A} \qquad \bar{y} = \frac{\int_D dA \ y}{A}. \tag{7}$$

   If the boundary of D is a piecewise smooth closed curve C, Green's theorem reads

$$\int_D dA \ \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right) = \oint_C \left(dx \ f(x,y) + dy \ g(x,y)\right), \tag{8}$$

   where $f(x, y)$ and $g(x, y)$ are function with continuous first derivatives and the curve is oriented anti-clockwise. Now taking $g(x, y) = \frac{1}{2}x^2$ and $f(x, y) = 0$ yields

$$\int_D dA \ x = \frac{1}{2}\int_C dy \ x^2. \tag{9}$$

Inserting this into the formula for $\bar{x}$ gives

$$\bar{x} = \frac{1}{2A} \oint_C dy x^2 \tag{10}$$

Similarly the choice $f(x, y) = \frac{1}{2}y^2$, $g(x, y) = 0$ gives

$$\bar{y} = -\frac{1}{2A} \oint_C dx\ y^2 \tag{11}$$

Here C comprises the semi-circular arc plus the line segment joining $(-1, 0)$ and $(1, 0)$. The integral $\int_C\ dy\ x^2$ is zero since the positive x part of the arc integral cancels the negative x part. Also the integral along the line segment is zero. This implies that $\bar{x} = 0$.

For the other integral, $\int_C\ dx\ y^2$, only the arc contributes since $y = 0$ along the line segment. Now

$$\int_C\ dx\ y^2 = -\int_{C,\text{clockwise}}\ dx\ y^2 = -\int_{-1}^{1}\ dx\ (1 - x^2) = -\frac{4}{3}. \tag{12}$$

Since $A = \frac{1}{2}\pi$ it follows that

$$\bar{y} = \frac{\frac{4}{3}}{2A} = \frac{4}{3\pi}. \tag{13}$$

Problem Sheet 8

1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (1-u^2)\mathbf{k} \tag{14}$$

with $1 \leq u \leq 2$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

Solution: So,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \cos v\mathbf{i} + \sin v\mathbf{j} - 2u\mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -u\sin v\mathbf{i} + u\cos v\mathbf{j} \end{aligned} \tag{15}$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 2u^2\cos v \\ 2u^2\sin v \\ u \end{pmatrix} \tag{16}$$

Now, on the parabola

$$\mathbf{F} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + \mathbf{k} \tag{17}$$

so, taking the dot product, the flux, ϕ, is

$$\begin{aligned} \phi &= \int_1^2 du \int_0^{2\pi} dv \left(2u^3\cos^2 v + 2u^3\sin^2 v + u^2\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned} \tag{18}$$

which is a positive answer, so the orientation was correct.

2. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{19}$$

with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

Solution: And again with the paraboloid:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -2\sin v\mathbf{i} + \cos v\mathbf{j} \end{aligned} \tag{20}$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -\cos v \\ -2\sin v \\ 0 \end{pmatrix} \tag{21}$$

On the parabola

$$\mathbf{F} = e^{-\sin v}\mathbf{i} - \sin v\mathbf{j} + 2\cos v \sin u\mathbf{k} \tag{22}$$

and the flux is

$$\phi = \int_0^5 du \int_0^{2\pi} dv \left(-\cos v e^{-sinv} + 2\sin^2 v\right) \tag{23}$$

Now,

$$\int_0^{2\pi} dv \cos v e^{-sinv} = 0 \tag{24}$$

there are lots of ways to see this, one way is to note that the integrand is odd about the point $v = \pi/2$ and a change of variable and the periodicity could be used to make the integral symmetric about this point

$$\int_0^{2\pi} dv \cos v e^{-sinv} = \int_{-\pi/2}^{3\pi/2} dv \cos v e^{-sinv} \tag{25}$$

and then let $w = v - \pi/2$. This leaves the other bit of the integral, which we do using the usual

$$2\sin^2 x = 1 - \cos 2x \tag{26}$$

giving

$$\phi = 10\phi \tag{27}$$

3. Use Green's Theorem to evaluate

$$\oint_c (y^2 dx + x^2 dy) \tag{28}$$

where C is the square with vertice $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and oriented anti-clockwise.

*Solution:*By Green's theorem

$$\oint_c (y^2 dx + x^2 dy) = \int_0^1 dx \int_0^1 dy(2x - 2y) = \int_0^1 dx(2x-1) = 0 \tag{29}$$

4. Calculate directly and using Stoke's Theorem

$$\int_S \mathbf{F}\cdot\mathbf{dS} \tag{30}$$

where $\mathbf{F} = (z-y)\mathbf{i} + (z+x)\mathbf{j} - (x+y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.

*Solution:*So, to calculate directly, choose some parameterization

$$\mathbf{r} = \rho\cos\phi\mathbf{i} + \rho\sin\phi\mathbf{j} + (9-\rho^2)\mathbf{k} \tag{31}$$

works. Now

$$\frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin \phi \mathbf{i} + \rho \cos \phi \mathbf{j} \tag{32}$$

$$frac\partial \mathbf{r} \partial \phi = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} - 2\rho \mathbf{k} \tag{33}$$

and, choosing the other order to make the normal upward points

$$\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 2\rho^2 \cos \phi \\ 2\rho^2 \sin \phi \\ -\rho \end{pmatrix} \tag{34}$$

Now, writing this as $(2\rho x, 2\rho y, \rho)$ and doing the dot product with $\mathbf{F}$ we are left with only terms which are linear in x or y and since the ϕ integral goes all the way around, we see the answer is zero. Next, using Stokes

$$\int_S \mathbf{F} \cdot \mathbf{dS} = \oint_c \operatorname{curl} \mathbf{F} \cdot \mathbf{dl} \tag{35}$$

Now, C is a circle of radius three, so $\mathbf{l} = 3\cos\phi \mathbf{i} + 3\sin\phi \mathbf{j}$ so $\mathbf{dl} = (-3sin\phi, 3\cos\phi, 0)$ and curl$\mathbf{F} = (-2, 2, 2)$ giving an integrand which is once again linear in sine and cosine and, once again, the answer zero.

Problem Sheet 9

1. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

   *Solution:*Let S be the closed surface comprising the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the disk (needed to close the surface) $z = 0$, $x^2 + y^2 \leq 1$. Using Gauss' theorem the flux of $\mathbf{F}$ out of S is

   $$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_D \text{div } \mathbf{F} \; dV = 3 \int_D (x^2 + y^2 + z^2) \; dV,$$

   where D is the region enclosed by S. This integral can be worked out through spherical polar coordinates or just by splitting D into small spherical half-shells of volume $2\pi r^2 \delta r$:

   $$\int_S \mathbf{F} \cdot d\mathbf{A} = 2\pi \int_0^1 dr \; r^2 \; 3r^2 = \frac{6\pi}{5}.$$

   Now the flux out of the disk is zero since here $\mathbf{F}$ is perpendicular to the outward normal $\mathbf{n} = -\mathbf{k}$. Thus the flux through the hemisphere is $6\pi/5$. The flux out of the whole sphere is $12\pi/5$.

2. Consider, again, the vector field

   $$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

   (a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.

   (b) Compute the flux of $\mathbf{F}$ out of the box $1 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

   (c) Compute the flux of $\mathbf{F}$ out of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

   Solution:

   (a) Flux integral trivial since $\mathbf{F} \cdot \mathbf{n}$ is constant over the sphere ($\mathbf{n}$ is the outward normal). Here $\mathbf{F} \cdot \mathbf{n} = 1/a^2$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{A} = 4\pi a^2 \; \frac{1}{a^2} = 4\pi$. Note that this is independent of the radius of the sphere.

   (b) We know div$\mathbf{F} = 0$. Therefore Gauss' theorem implies that the flux out of the box is zero. Note that this result is very difficult to obtain via a direct surface integral computation.

   (c) Naively, Gauss' theorem also gives a zero flux in this case. However, this is not correct. Indeed, an indiscriminate use of Gauss' theorem would also give a zero flux in part i). The point is that in parts i) and iii) (but not ii)) the 'inside' region contains the origin where $\mathbf{F}$ is singular. As in part i) the correct answer to this question is 4π. This can be obtained by combining Gauss' theorem with the result of the direct calculation of part a). To do this remove a sphere (centred at the origin) of radius less than one from the box. In this region $\mathbf{F}$ is smooth and so Gauss' theorem implies that the flux out of this region is zero. However this flux comprises two parts; the flux out of the box (which we wish to compute) and the flux into the cut sphere (which from part i) is -4π). Therefore the flux out of the box must be 4π.

3. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

   Solution: Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 \, dt \; \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$. Now $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k} = \mathbf{r} - 3z\mathbf{k}$ so that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 \, dt \; -3zt\mathbf{k} \times \mathbf{r}t = -3\int_0^1 \, dt \; t^2 \; (zx\mathbf{j} - zy\mathbf{i}) = zy\mathbf{i} - zx\mathbf{j}.$$

4. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = e^x\mathbf{k}$.

   Solution: $\mathbf{A} = e^x\mathbf{j}$ by inspection. Using the formula actually gives a different vector potential, this is possible because the vector potential is only defined up to an irrotational field,

$$\mathbf{A}(\mathbf{r}) = \left(e^x + \frac{(1-e^x)}{x}\right)\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i}.$$

   The two vector potentials differ by a gradient

$$\mathbf{A}_{II} - \mathbf{A}_I = \frac{1-e^x}{x}\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i} = \nabla\phi.$$

   where

$$\phi = y\frac{1-e^x}{x}.$$

5. Find a Hodge decomposition for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

   Solution: So the Hodge decomposition is $\mathbf{F} = \nabla\phi + curl\,\mathbf{A}$ which implies $\triangle\phi = 1$. A convenient choice here is

$$\phi = \frac{1}{2}z^2$$

   leaving curl$\mathbf{A} = -y\mathbf{i} + x\mathbf{j}$ but we have looked at examples like this before, $-y\mathbf{i} + x\mathbf{j} = \mathbf{k} \times \mathbf{r}$ so $\mathbf{A} = 2\mathbf{k}$.