

231 Outline Solutions Tutorial Sheet 7, 8 and 9.¹²

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Problem Sheet 7

1. Which of the following vector fields are conservative?

- (a) $\mathbf{F} = -yz \sin x \mathbf{i} + z \cos x \mathbf{j} + y \cos x \mathbf{k}$.
- (b) $\mathbf{F} = \frac{1}{2}y \mathbf{i} - \frac{1}{2}x \mathbf{j}$.
- (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ where \mathbf{B} is a constant vector.

Solution:

- (a) $\mathbf{F} = \nabla yz \cos x$ so \mathbf{F} is conservative.
 - (b) $\text{curl } \mathbf{F} = \mathbf{k} \neq 0$ so \mathbf{F} is not conservative.
 - (c) A short calculation gives $\text{curl } \mathbf{F} = \mathbf{B}$ so \mathbf{F} is not conservative. Remark: $\frac{1}{2}(\mathbf{B} \times \mathbf{r})$ is a vector potential for the constant vector field \mathbf{B} .
2. Compute the flux of the vector field $\mathbf{F} = (x + x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \leq z \leq 1$.

*Solution:*As we have seen before, $\mathbf{r}(u, v) = \cos u \mathbf{i} + \sin u \mathbf{j} + v \mathbf{k}$ giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos u \mathbf{i} + \sin u \mathbf{j}. \quad (1)$$

and

$$\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (x + x^2) \cos u + y \sin u = \cos^2 + \cos^3 u + \sin^2 u = 1 + \cos^3 u. \quad (2)$$

and so, noting that the \mathbf{F} is perpendicular to the normal at both ends and so we need only include the curved surface

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_0^1 dv \int_0^{2\pi} du (1 + \cos^3 u) = 2\pi, \quad (3)$$

since the $\cos^3 u$ integral is zero by symmetry.

Note: This problem can be solved by noting that $x^2\mathbf{i}$ makes no contribution (by symmetry) and $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has flux 2π since $\mathbf{F} \cdot \mathbf{n} = 1$.

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3. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the $z = 0$ plane.

*Solution:*To parametrize the sphere choose $(u, v) = (\theta, \phi)$, that is, spherical polar angles. Since the radius $r = a$ this gives $x(u, v) = a \sin u \cos v$, $y(u, v) = a \sin u \sin v$, $z(u, v) = a \cos u$ with $0 \leq \theta \leq \pi/2$ and $0 \leq \phi < 2\pi$. Since \mathbf{F} only has a non-zero z -component we just need

$$\begin{aligned} \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right)_3 &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \\ &= a^2 (\cos u \cos v \sin u \cos v + \cos u \sin v \sin u \sin v) \\ &= a^2 \cos u \sin u, \end{aligned} \quad (4)$$

which is positive. So the orientation is upwards. Now $F_z = z^3 = a^3 \cos^3 u$, and so

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{A} &= a^5 \int_0^{\frac{1}{2}\pi} du \int_0^{2\pi} dv \cos^4 u \sin u \\ &= 2\pi a^5 \int_0^{\frac{1}{2}\pi} du \cos^4 u \sin u \\ &= 2\pi a^5 \left[-\frac{\cos^5 u}{5} \right]_0^{\frac{1}{2}\pi} = \frac{2\pi a^5}{5}. \end{aligned} \quad (5)$$

4. Let D be a plane region with area A whose boundary is a piecewise smooth closed curve C . Use Green's theorem to prove that the centroid (\bar{x}, \bar{y}) of D is

$$\begin{aligned} \bar{x} &= \frac{1}{2A} \oint_C dy x^2 \\ \bar{y} &= -\frac{1}{2A} \oint_C dx y^2. \end{aligned} \quad (6)$$

Use this result to compute the centroid of a semi-circle (this was determined in the lectures using the more standard formula).

*Solution:*The centroid (\bar{x}, \bar{y}) of a plane region D is given by

$$\bar{x} = \frac{\int_D dA x}{A} \quad \bar{y} = \frac{\int_D dA y}{A}. \quad (7)$$

If the boundary of D is a piecewise smooth closed curve C , Green's theorem reads

$$\int_D dA \left(\frac{\partial g(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y} \right) = \oint_C (dx f(x, y) + dy g(x, y)), \quad (8)$$

where $f(x, y)$ and $g(x, y)$ are function with continuous first derivatives and the curve is oriented anti-clockwise. Now taking $g(x, y) = \frac{1}{2}x^2$ and $f(x, y) = 0$ yields

$$\int_D dA x = \frac{1}{2} \int_C dy x^2. \quad (9)$$

Inserting this into the formula for \bar{x} gives

$$\bar{x} = \frac{1}{2A} \oint_C dy x^2 \quad (10)$$

Similarly the choice $f(x, y) = \frac{1}{2}y^2$, $g(x, y) = 0$ gives

$$\bar{y} = -\frac{1}{2A} \oint_C dx y^2 \quad (11)$$

Here C comprises the semi-circular arc plus the line segment joining $(-1, 0)$ and $(1, 0)$. The integral $\int_C dy x^2$ is zero since the positive x part of the arc integral cancels the negative x part. Also the integral along the line segment is zero. This implies that $\bar{x} = 0$.

For the other integral, $\int_C dx y^2$, only the arc contributes since $y = 0$ along the line segment. Now

$$\int_C dx y^2 = -\int_{C, \text{clockwise}} dx y^2 = -\int_{-1}^1 dx (1 - x^2) = -\frac{4}{3}. \quad (12)$$

Since $A = \frac{1}{2}\pi$ it follows that

$$\bar{y} = \frac{\frac{4}{3}}{2A} = \frac{4}{3\pi}. \quad (13)$$

Problem Sheet 8

1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + (1 - u^2)\mathbf{k} \quad (14)$$

with $1 \leq u \leq 2$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

Solution: So,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \cos v \mathbf{i} + \sin v \mathbf{j} - 2u \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -u \sin v \mathbf{i} + u \cos v \mathbf{j} \end{aligned} \quad (15)$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 2u^2 \cos v \\ 2u^2 \sin v \\ u \end{pmatrix} \quad (16)$$

Now, on the paraboloid

$$\mathbf{F} = u \cos v \mathbf{i} + u \sin v \mathbf{j} + \mathbf{k} \quad (17)$$

so, taking the dot product, the flux, ϕ , is

$$\begin{aligned} \phi &= \int_1^2 du \int_0^{2\pi} dv (2u^3 \cos^2 v + 2u^3 \sin^2 v + u^2) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned} \quad (18)$$

which is a positive answer, so the orientation was correct.

2. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x \sin z \mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u, v) = 2 \cos v \mathbf{i} + \sin v \mathbf{j} + u \mathbf{k} \quad (19)$$

with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

Solution: And again with the paraboloid:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -2 \sin v \mathbf{i} + \cos v \mathbf{j} \end{aligned} \quad (20)$$

giving

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -\cos v \\ -2 \sin v \\ 0 \end{pmatrix} \quad (21)$$

On the parabola

$$\mathbf{F} = e^{-\sin v} \mathbf{i} - \sin v \mathbf{j} + 2 \cos v \sin u \mathbf{k} \quad (22)$$

and the flux is

$$\phi = \int_0^5 du \int_0^{2\pi} dv (-\cos v e^{-\sin v} + 2 \sin^2 v) \quad (23)$$

Now,

$$\int_0^{2\pi} dv \cos v e^{-\sin v} = 0 \quad (24)$$

there are lots of ways to see this, one way is to note that the integrand is odd about the point $v = \pi/2$ and a change of variable and the periodicity could be used to make the integral symmetric about this point

$$\int_0^{2\pi} dv \cos v e^{-\sin v} = \int_{-\pi/2}^{3\pi/2} dv \cos v e^{-\sin v} \quad (25)$$

and then let $w = v - \pi/2$. This leaves the other bit of the integral, which we do using the usual

$$2 \sin^2 x = 1 - \cos 2x \quad (26)$$

giving

$$\phi = 10\phi \quad (27)$$

3. Use Green's Theorem to evaluate

$$\oint_C (y^2 dx + x^2 dy) \quad (28)$$

where C is the square with vertex $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and oriented anti-clockwise.

Solution: By Green's theorem

$$\oint_C (y^2 dx + x^2 dy) = \int_0^1 dx \int_0^1 dy (2x - 2y) = \int_0^1 dx (2x - 1) = 0 \quad (29)$$

4. Calculate directly and using Stoke's Theorem

$$\int_S \mathbf{F} \cdot d\mathbf{S} \quad (30)$$

where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.

Solution: So, to calculate directly, choose some parameterization

$$\mathbf{r} = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + (9 - \rho^2) \mathbf{k} \quad (31)$$

works. Now

$$\frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin \phi \mathbf{i} + \rho \cos \phi \mathbf{j} \quad (32)$$

$$\frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} - 2\rho \mathbf{k} \quad (33)$$

and, choosing the other order to make the normal upward points

$$\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 2\rho^2 \cos \phi \\ 2\rho^2 \sin \phi \\ -\rho \end{pmatrix} \quad (34)$$

Now, writing this as $(2\rho x, 2\rho y, \rho)$ and doing the dot product with \mathbf{F} we are left with only terms which are linear in x or y and since the ϕ integral goes all the way around, we see the answer is zero. Next, using Stokes

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{l} \quad (35)$$

Now, C is a circle of radius three, so $\mathbf{l} = 3 \cos \phi \mathbf{i} + 3 \sin \phi \mathbf{j}$ so $d\mathbf{l} = (-3 \sin \phi, 3 \cos \phi, 0)$ and $\text{curl} \mathbf{F} = (-2, 2, 2)$ giving an integrand which is once again linear in sine and cosine and, once again, the answer zero.

Problem Sheet 9

- Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

Solution: Let S be the closed surface comprising the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and the disk (needed to close the surface) $z = 0, x^2 + y^2 \leq 1$. Using Gauss' theorem the flux of \mathbf{F} out of S is

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_D \operatorname{div} \mathbf{F} dV = 3 \int_D (x^2 + y^2 + z^2) dV,$$

where D is the region enclosed by S . This integral can be worked out through spherical polar coordinates or just by splitting D into small spherical half-shells of volume $2\pi r^2 \delta r$:

$$\int_S \mathbf{F} \cdot d\mathbf{A} = 2\pi \int_0^1 dr r^2 3r^2 = \frac{6\pi}{5}.$$

Now the flux out of the disk is zero since here \mathbf{F} is perpendicular to the outward normal $\mathbf{n} = -\mathbf{k}$. Thus the flux through the hemisphere is $6\pi/5$. The flux out of the whole sphere is $12\pi/5$.

- Consider, again, the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

- Compute the flux of \mathbf{F} out of a sphere of radius a centred at the origin.
- Compute the flux of \mathbf{F} out of the box $1 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1$.
- Compute the flux of \mathbf{F} out of the box $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$.

Solution:

- Flux integral trivial since $\mathbf{F} \cdot \mathbf{n}$ is constant over the sphere (\mathbf{n} is the outward normal). Here $\mathbf{F} \cdot \mathbf{n} = 1/a^2$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{A} = 4\pi a^2 \frac{1}{a^2} = 4\pi$. Note that this is independent of the radius of the sphere.
- We know $\operatorname{div} \mathbf{F} = 0$. Therefore Gauss' theorem implies that the flux out of the box is zero. Note that this result is very difficult to obtain via a direct surface integral computation.
- Naively, Gauss' theorem also gives a zero flux in this case. However, this is not correct. Indeed, an indiscriminate use of Gauss' theorem would also give a zero flux in part i). The point is that in parts i) and iii) (but not ii)) the 'inside' region contains the origin where \mathbf{F} is singular. As in part i) the correct answer to this question is 4π . This can be obtained by combining Gauss' theorem with the result of the direct calculation of part a). To do this remove a sphere (centred at

the origin) of radius less than one from the box. In this region \mathbf{F} is smooth and so Gauss' theorem implies that the flux out of this region is zero. However this flux comprises two parts; the flux out of the box (which we wish to compute) and the flux into the cut sphere (which from part i) is -4π). Therefore the flux out of the box must be 4π .

- Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

Solution: Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 dt \mathbf{F}(t\mathbf{r}) \times t\mathbf{r}$. Now $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k} = \mathbf{r} - 3z\mathbf{k}$ so that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 dt (-3zt\mathbf{k} \times t\mathbf{r}) = -3 \int_0^1 dt t^2 (zx\mathbf{j} - zy\mathbf{i}) = zy\mathbf{i} - zx\mathbf{j}.$$

- Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = e^x\mathbf{k}$.

Solution: $\mathbf{A} = e^x\mathbf{j}$ by inspection. Using the formula actually gives a different vector potential, this is possible because the vector potential is only defined up to an irrotational field,

$$\mathbf{A}(\mathbf{r}) = \left(e^x + \frac{(1 - e^x)}{x} \right) \mathbf{j} - y \left(\frac{e^x}{x} - \frac{(e^x - 1)}{x^2} \right) \mathbf{i}.$$

The two vector potentials differ by a gradient

$$\mathbf{A}_{II} - \mathbf{A}_I = \frac{1 - e^x}{x} \mathbf{j} - y \left(\frac{e^x}{x} - \frac{(e^x - 1)}{x^2} \right) \mathbf{i} = \nabla \phi.$$

where

$$\phi = y \frac{1 - e^x}{x}.$$

- Find a Hodge decomposition for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

Solution: So the Hodge decomposition is $\mathbf{F} = \nabla \phi + \operatorname{curl} \mathbf{A}$ which implies $\Delta \phi = 1$. A convenient choice here is

$$\phi = \frac{1}{2}z^2$$

leaving $\operatorname{curl} \mathbf{A} = -y\mathbf{i} + x\mathbf{j}$ but we have looked at examples like this before, $-y\mathbf{i} + x\mathbf{j} = \mathbf{k} \times \mathbf{r}$ so $\mathbf{A} = 2\mathbf{k}$.