231 Outline Solutions Tutorial Sheet 4, 5 and 6.[1][2]

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Problem Sheet 4

1. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

is divergenceless.

*Solution:*So

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} \quad (1)$$

Using the product rule

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5} \quad (2)$$

and so

$$\nabla \cdot \mathbf{F} = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0 \quad (3)$$

using $r^2 = x^2 + y^2 + z^2$. Note by the way we have used

$$\frac{\partial}{\partial x} r = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r} \quad (4)$$

using the chain rule.

2. Show $\operatorname{div}\mathbf{r} = 3$ and $\operatorname{grad}|\mathbf{r}| = \mathbf{r}/|\mathbf{r}|$.

*Solution:*Well

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (5)$$

and so

$$\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3 \quad (6)$$

and

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k} \quad (7)$$

and

$$\frac{\partial}{\partial x} r = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r} \quad (8)$$

by the chain rule.

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[2]Including material from Chris Ford, to whom many thanks.

3. Find $\nabla(1/|\mathbf{r}|)$.

   *Solution:*So this is similar to the previous one

$$\nabla \frac{1}{r} = \frac{\partial}{\partial x}\frac{1}{r}\mathbf{i} + \frac{\partial}{\partial y}\frac{1}{r}\mathbf{j} + \frac{\partial}{\partial z}\frac{1}{r}\mathbf{k} \tag{9}$$

   and

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial}{\partial x}r = -\frac{x}{r^3} \tag{10}$$

   hence

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3} \tag{11}$$

   and this, or course, is consistient with $\text{curl}\,\mathbf{r}/r^3 = 0$.

4. Show $\text{grad}\, f(r) = f'(r)\hat{\mathbf{r}}$ where $r = |\mathbf{r}|$. If $\mathbf{F}(r) = f(r)\mathbf{r}$ find $\text{div}\mathbf{F}(r)$. Find $\text{div}\,\text{grad}\, f(r)$.

   *Solution:*So, now, we use the chain rule to show

$$\frac{\partial}{\partial x}f(r) = f'(r)\frac{\partial r}{\partial x} = \frac{xf'(r)}{r} \tag{12}$$

   and, since the gradient has three terms of this form, it is easy to see $\text{grad}\, f(r) = f'(r)\hat{\mathbf{r}}$. As for the divergence, $F_1 = xf(r)$ and

$$\frac{\partial}{\partial x}xf(r) = f(r) + \frac{x^2 f'}{r} \tag{13}$$

   and so, adding three similar terms together, we get

$$\nabla \cdot \mathbf{F} = 3f + rf' \tag{14}$$

   Finally, , we know the $\text{grad}\, f(r)$ and, so, using f'/r for f in the divergence formula we get

$$\triangle f(r) = \frac{3f'}{r} + r\left(\frac{f'}{r}\right)' = \frac{2f'}{r} + f'' \tag{15}$$

   which gives us a formula for the laplacian of a spherically symmetric field in polar coördinates, later we will see how to convert partial differential equations from one coördinate system to another.

Problem Sheet 5

1. Show that away from the origin the vector field
$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{16}$$
is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$).

   *Solution:*Note that $\mathbf{F} = \text{grad}\,(-1/r)$ and so curl $\mathbf{F} = 0$. This can also be done by direct calculation.

2. Calculate $\text{curl}\,\mathbf{r}/r$ and $\text{div}\,\mathbf{r}/r$ away from the origin. What is $\triangle r$?

   *Solution:*So it is easy enough to check these by hand, for example, the bfi component of $\text{curl}\,\mathbf{r}/r$ is given by
$$\left(\nabla \times \frac{\mathbf{r}}{r}\right)_1 = \frac{\partial}{\partial y}\frac{z}{r} - \frac{\partial}{\partial z}\frac{y}{r} = 0 \tag{17}$$
and, with the other two components similar $\text{curl}\,\mathbf{r}/r = 0$. As for the divergence
$$\frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} \tag{18}$$
and hence $\text{div}\,\mathbf{r}/r = 2/r$. Finally, $\text{grad}\,r = bfr/r$ so $\triangle r = 2/r$.

3. Prove the identity
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{19}$$

   *Solution:*So this is easy by direct calculation,
$$\nabla \cdot (\nabla \times \mathbf{F}) = \partial_x(\partial_y F_3 - \partial_z F_2) + \partial_y(\partial_z F_1 - \partial_x F_3) + \partial_z(\partial_x F_2 - \partial_y F_1) \tag{20}$$
and expanding out, all the terms cancel, assuming the partial derivative commute.

4. Prove the identity
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \triangle \mathbf{F}. \tag{21}$$

   *Solution:*Lets do the first component:
$$[\nabla \times (\nabla \times \mathbf{F})]_1 = \frac{\partial}{\partial y}(F_{2,x} - F_{1,y}) - \frac{\partial}{\partial z}(F_{3,x} - F_{1,z}) \tag{22}$$
where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and I am using a comma notation for differenciation so for example
$$F_{2,x} = \frac{\partial F_2}{\partial x} \tag{23}$$
Now, taking away some brackets
$$[\nabla \times (\nabla \times \mathbf{F})]_1 = F_{2,xy} - F_{1,yy} - F_{3,xz} - F_{1,zz} \tag{24}$$

Coming from the other side

$$[\nabla(\nabla\cdot\mathbf{F})]_1 = \frac{\partial}{\partial x}(F_{1,x}+F_{2,y}+F_{3,z}) = F_{1,xx}+F_{2,yx}+F_{3,zx} \tag{25}$$

so

$$[\nabla\times(\nabla\times\mathbf{F})]_1 - [\nabla(\nabla\cdot\mathbf{F})]_1 = F_{1,xx}+F_{1,yy}+F_{1,zz} = [\triangle\mathbf{F}]_1 \tag{26}$$

and similarily for the other components.

5. Compute the line integrals:

   (a) $\int_C\ (dx\ xy\ +\frac{1}{2}dy\ x^2\ +dz)$ where C is the line segment joining the origin and the point $(1,1,2)$.

   (b) $\int_C\ (dx\ yz\ +dy\ xz\ +dz\ yx^2)$ where C is the same line as in the previous part

   *Solution:*A quick way here is to note that $\mathbf{F}$ is conservative.

$$\mathbf{F} = xy\mathbf{i}+\frac{1}{2}x^2\mathbf{j}+\mathbf{k} = \nabla\phi \tag{27}$$

where $\phi = \frac{1}{2}x^2y+z$. Hence

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \phi(1,1,2)-\phi(0,0,0) = \frac{5}{2}. \tag{28}$$

For the next part, use the parametrization $x(u)=u$, $y(u)=u$, $z(u)=2u$ $(0\le u\le 1)$.

$$\begin{aligned}\frac{d\mathbf{r}}{du} &= \mathbf{i}+\mathbf{j}+2\mathbf{k},\\ \mathbf{F}\cdot\frac{d\mathbf{r}}{du} &= 2u^2+2u^2+2u^3 = 4u^2+2u^3\end{aligned} \tag{29}$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 du\ \left(4u^2+2u^3\right) = \frac{4}{3}+\frac{1}{2} = \frac{11}{6}. \tag{30}$$

Problem Sheet 6

1. For each of the following vector fields compute the line integral $\oint_C \mathbf{F}\cdot\mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2y\mathbf{j}$.

   *Solution:*In the first part $\mathbf{F} = \nabla\frac{1}{2}(x^2+y^2)$ so that $\mathbf{F}$ is conservative giving $\oint_C \mathbf{F}\cdot\mathbf{dl}$. In the second part parametrize curve:

$$\begin{aligned} x(u) &= \cos u \\ y(u) &= \sin u \\ z(u) &= 0 \end{aligned} \tag{31}$$

   where $0 \le u \le 2\pi$ or $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}$. Now

$$\frac{d\mathbf{r}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j}. \tag{32}$$

   and

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{du} = -y\sin u - x^2y\cos u = -\sin^2 u - \cos^3 u\sin u. \tag{33}$$

   Thus

$$\oint_C \mathbf{F}\cdot\mathbf{dl} = \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^3 u\sin u\right) = -\pi, \tag{34}$$

   since the average value of $\sin^2 u$ is $\frac{1}{2}$ and $\int_0^{2\pi} du\ \cos^3 u\sin u = 0$ by symmetry.

2. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla\phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = x^2y\mathbf{i} + 5xy^2\mathbf{j}$

   (c) $\mathbf{F} = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$

   (d) $\mathbf{F} = x\log y\mathbf{i} + y\log x\mathbf{j}$

   *Solution:*So, in the first case we know $\mathbf{F}$ is conservative, it was already discussed in the last question, here we go again, it is easy to see the curl is zero, having done that we want $\mathbf{F} = \nabla\phi$, hence $F_1 = \phi_{,x}$ or

$$\frac{\partial}{\partial x}\phi = x \tag{35}$$

   and hence $\phi = x^2/2 + C(y,z)$, where $C(y,z)$ is an arbitrary function of y and z, substitute that back in to get

$$\frac{\partial}{\partial y}C = y \tag{36}$$

giving $\phi = x^2/2 + y^2/2 + C(z)$ where $C(z) = C$ a constant follows from $F_3 = 0$. For the next one it is easy to calculate that $\nabla \times \mathbf{F} = (5y^2 - x^2)\mathbf{k}$. The next one once again has a zero curl, so, $\mathbf{F} = \nabla\phi$ and so

$$\frac{\partial}{\partial x}\phi = e^x \cos y \tag{37}$$

Hence $\phi = \exp x \cos y + C(y, z)$. Substituting this into the y equation shows $C(y, z) = C(z)$ and finally, substituting this into the z equation, we get $C(z) = C$, a constant. The last one has a nonzero curl

$$\nabla \times \mathbf{F} = \left(\frac{y}{x} - \frac{x}{y}\right)\mathbf{k} \tag{38}$$

and isn't conservative.

3. If C is a straight line from (x', y, z) to (x, y, z) show

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_{x'}^{x} F_1 dx \tag{39}$$

*Solution:*So, here, using the usual formula for a straight line from $\mathbf{a}$ to $\mathbf{b}$, $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

$$\mathbf{r} = (x' + t(x - x'))\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{40}$$

and, as a consequence

$$\frac{d\mathbf{r}}{dt} = (x - x')\mathbf{i} \tag{41}$$

we get

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 F_1(x' + t(x - x'), y, z)(x - x')dt \tag{42}$$

and, let $\tilde{x} = x' + t(x - x')$ before renaming $\tilde{x}$ to x.

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}.$$

Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.

Solution:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \frac{1}{2}\mathbf{k}\left[\partial_x\left(\frac{-x}{x^2 + y^2}\right) + \partial_y\left(\frac{y}{x^2 + y^2}\right)\right] \\
&= \frac{1}{2}\mathbf{k}\left[-\frac{1}{x^2 + y^2} + \frac{2x^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2}\right]
\end{aligned}$$

$$= \quad 0. \tag{43}$$

To show that $\mathbf{F}$ is not conservative consider $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle. Using the obvious parametrization

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{dl} &= \int_0^{2\pi} du \, \left(-\sin^2 u - \cos^2 u \right) \\ &= -2\pi \neq 0, \end{aligned} \tag{44}$$

therefore $\mathbf{F}$ is not conservative.

The domain $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$ is simply connected and $\mathbf{F}$ is irrotational and smooth is the domain. Thus $\mathbf{F}$ is conservative.

Write $\mathbf{F} = \nabla\phi$. Seek a $\phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = \frac{y}{x^2 + y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{x}{x^2 + y^2}. \tag{45}$$

Integrate first equation by treating y as a constant

$$\phi(x, y) = y \int \frac{dx}{x^2 + y^2} = \tan^{-1} \frac{x}{y} + C(y). \tag{46}$$

Assume that x and y are non-negative, then

$$\tan^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x} = \frac{\pi}{2},$$

so that $\phi(x, y) = -\tan^{-1} \frac{y}{x} +$ a possibly y-dependent constant. However it is easy to check that $\phi = -\tan^{-1} \frac{y}{x}$ satisfies $\frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}$. Clearly, $\tan^{-1} \frac{y}{x}$ is the usual polar angle θ, that is $\phi = -\theta$.

Can try to extend this back to the original domain $x^2 + y^2 \geq \frac{1}{2}$, but ϕ will suffer a branch cut discontinuity at, say $\theta = \frac{3}{2}\pi$.