231 Outline Solutions Tutorial Sheet 4, 5 and 6.12

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Problem Sheet 4

1. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

is divergenceless.

Solution:So

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} \frac{x}{r^3} + \frac{\partial}{\partial y} \frac{y}{r^3} + \frac{\partial}{\partial y} \frac{y}{r^3} \tag{1}$$

Using the product rule

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$
(2)

and so

$$\nabla \cdot \mathbf{F} = \frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} = 0 \tag{3}$$

using $r^2 = x^2 + y^2 + z^2$. Note by the way we have used

$$\frac{\partial}{\partial x}r = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \tag{4}$$

using the chain rule.

2. Show div $\mathbf{r} = 3$ and grad $|\mathbf{r}| = \mathbf{r}/|\mathbf{r}|$.

Solution:Well

 $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{5}$

(6)

and so

$$\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$$

and

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}$$
(7)

and

$$\frac{\partial}{\partial x}r = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \tag{8}$$

by the chain rule.

¹Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231 ²Including material from Chris Ford, to whom many thanks. 3. Find $\nabla(1/|\mathbf{r}|)$.

Solution: So this is similar to the previous one

$$\nabla \frac{1}{r} = \frac{\partial}{\partial x} \frac{1}{r} \mathbf{i} + \frac{\partial}{\partial y} \frac{1}{r} \mathbf{j} + \frac{\partial}{\partial z} \frac{1}{r} \mathbf{k}$$
(9)

and

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial}{\partial x}r = -\frac{x}{r^3} \tag{10}$$

hence

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3} \tag{11}$$

and this, or course, is consistient with $\operatorname{curl} \mathbf{r}/r^3 = 0$.

4. Show grad $f(r) = f'(r)\hat{\mathbf{r}}$ where $r = |\mathbf{r}|$. If $\mathbf{F}(r) = f(r)\mathbf{r}$ find div $\mathbf{F}(r)$. Find div grad f(r).

Solution: So, now, we use the chain rule to show

$$\frac{\partial}{\partial x}f(r) = f'(r)\frac{\partial r}{\partial x} = \frac{xf'(r)}{r}$$
(12)

and, since the gradient has three terms of this form, it is easy to see grad $f(r) = f'(r)\hat{\mathbf{r}}$. As for the divergence, $F_1 = xf(r)$ and

$$\frac{\partial}{\partial x}xf(r) = f(r) + \frac{x^2f'}{r} \tag{13}$$

and so, adding three similar terms together, we get

$$\nabla \cdot \mathbf{F} = 3f + rf' \tag{14}$$

Finally, , we know the grad f(r) and, so, using f^\prime/r for f in the divergence formula we get

$$\Delta f(r) = \frac{3f'}{r} + r\left(\frac{f'}{r}\right)' = \frac{2f'}{r} + f'' \tag{15}$$

which gives us a formula for the laplacian of a spherically symmetric field in polar coördinates, later we will see how to convert partial differential equations from one coördinate system to another.

Problem Sheet 5

1. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{16}$$

is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$). Solution:Note that $\mathbf{F} = \text{grad}(-1/r)$ and so curl $\mathbf{F} = 0$. This can also be done by direct calculation.

2. Calculate curl \mathbf{r}/r and div \mathbf{r}/r away from the origin. What is Δr ?

Solution: So it is easy enough to check these by hand, for example, the bfi component of ${\rm curl}\,{\bf r}/r$ is given by

$$\left(\nabla \times \frac{\mathbf{r}}{r}\right)_1 = \frac{\partial}{\partial y} \frac{z}{r} - \frac{\partial}{\partial z} \frac{y}{r} = 0 \tag{17}$$

and, with the other two components similar curl $\mathbf{r}/r = 0$. As for the divergence

$$\frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} \tag{18}$$

and hence div $\mathbf{r}/r = 2/r$. Finally, grad r = bfr/r so $\Delta r = 2/r$.

3. Prove the identity

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{19}$$

Solution: So this is easy by direct calculation,

$$\nabla \cdot (\nabla \times \mathbf{F}) = \partial_x (\partial_y F_3 - \partial_z F_2) + \partial_y (\partial_z F_1 - \partial_x F_3) + \partial_z (\partial_x F_2 - \partial_y F_1)$$
(20)

and expanding out, all the terms cancel, assuming the partial derivative commute.

4. Prove the identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \Delta \mathbf{F}.$$
(21)

Solution:Lets do the first component:

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = \frac{\partial}{\partial y} (F_{2,x} - F_{1,y}) - \frac{\partial}{\partial z} (F_{3,x} - F_{1,z})$$
(22)

where $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ and I am using a comma notation for differenciation so for example

$$F_{2,x} = \frac{\partial F_2}{\partial x} \tag{23}$$

Now, taking away some brackets

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = F_{2,xy} - F_{1,yy} - F_{3,xz} - F_{1,zz}$$
(24)

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Coming from the other side

$$[\nabla(\nabla \cdot \mathbf{F})]_1 = \frac{\partial}{\partial x} (F_{1,x} + F_{2,y} + F_{3,z}) = F_{1,xx} + F_{2,yx} + F_{3,zx}$$
(25)

 \mathbf{SO}

$$[\nabla \times (\nabla \times \mathbf{F})]_1 - [\nabla (\nabla \cdot \mathbf{F})]_1 = F_{1,xx} + F_{1,yy} + F_{1,zz} = [\Delta \mathbf{F}]_1$$
(26)

and similarly for the other components.

 \mathbf{F}

- 5. Compute the line integrals:
 - (a) $\int_C (dx xy + \frac{1}{2}dy x^2 + dz)$ where C is the line segment joining the origin and the point (1, 1, 2).
 - (b) $\int_C (dx yz + dy xz + dz yx^2)$ where C is the same line as in the previous part

Solution: A quick way here is to note that \mathbf{F} is conservative.

$$\mathbf{F} = xy\mathbf{i} + \frac{1}{2}x^2\mathbf{j} + \mathbf{k} = \nabla\phi \tag{27}$$

where $\phi = \frac{1}{2}x^2y + z$. Hence

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \phi(1, 1, 2) - \phi(0, 0, 0) = \frac{5}{2}.$$
 (28)

For the next part, use the parametrization x(u) = u, y(u) = u, z(u) = 2u $(0 \le u \le 1)$.

$$\frac{d\mathbf{r}}{du} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

$$\cdot \frac{d\mathbf{r}}{du} = 2u^2 + 2u^2 + 2u^3 = 4u^2 + 2u^3$$
(29)

so

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_0^1 du \left(4u^2 + 2u^3\right) = \frac{4}{3} + \frac{1}{2} = \frac{11}{6}.$$
(30)

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Problem Sheet 6

- 1. For each of the following vector fields compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.
 - (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$
 - (b) $\mathbf{F} = y\mathbf{i} x^2y\mathbf{j}$.

Solution: In the first part $\mathbf{F} = \nabla \frac{1}{2}(x^2 + y^2)$ so that \mathbf{F} is conservative giving $\oint_C \mathbf{F} \cdot \mathbf{dl}$. In the second part parametrize curve:

$$\begin{aligned} x(u) &= \cos u \\ y(u) &= \sin u \\ z(u) &= 0 \end{aligned}$$
 (31)

where $0 \le u \le 2\pi$ or $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}$. Now

$$\frac{d\mathbf{r}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j}.$$
(32)

and

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{du} = -y\sin u - x^2 y\cos u = -\sin^2 u - \cos^3 u\sin u.$$
(33)

Thus

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \int_0^{2\pi} du \ \left(-\sin^2 u - \cos^3 u \sin u \right) = -\pi, \tag{34}$$

since the average value of $\sin^2 u$ is $\frac{1}{2}$ and $\int_0^{2\pi} du \cos^3 u \sin u = 0$ by symmetry.

- 2. For each of these fields determine if **F** is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla \phi$.
 - (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ (b) $\mathbf{F} = x^2y\mathbf{i} + 5xy^2\mathbf{j}$ (c) $\mathbf{F} = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$ (d) $\mathbf{F} = x \log y\mathbf{i} + y \log x\mathbf{j}$

Solution: So, in the first case we know \mathbf{F} is conservative, it was already discussed in the last question, here we go again, it is easy to see the curl is zero, having done that we want $\mathbf{F} = \nabla \phi$, hence $F_1 = \phi_x$ or

$$\frac{\partial}{\partial x}\phi = x \tag{35}$$

and hence $\phi = x^2/2 + C(y, z)$, where C(y, z) is an arbitrary function of y and z, substitute that back in to get

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$$\frac{\partial}{\partial y}C = y \tag{36}$$

giving $\phi = x^2/2 + y^2/2 + C(z)$ where C(z) = C a constant follows from $F_3 = 0$. For the next one it is easy to calculate that $\nabla \times \mathbf{F} = (5y^2 - x^2)\mathbf{k}$. The next one once again has a zero curl, so, $\mathbf{F} = \nabla \phi$ and so

$$\frac{\partial}{\partial x}\phi = e^x \cos y \tag{37}$$

Hence $\phi = \exp x \cos y + C(y, z)$. Substituting this into the y equation shows C(y, z) = C(z) and finally, substituting this into the z equation, we get C(z) = C, a constant. The last one has a nonzero curl

$$\nabla \times \mathbf{F} = \left(\frac{y}{x} - \frac{x}{y}\right) \mathbf{k} \tag{38}$$

and isn't conservative.

3. If C is a straight line from (x', y, z) to (x, y, z) show

$$\int_C \mathbf{F} \cdot \mathbf{dl} = \int_{x'}^x F_1 dx \tag{39}$$

Solution: So, here, using the usual formula for a straight line from \mathbf{a} to \mathbf{b} , $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

$$\mathbf{r} = (x' + t(x - x'))\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
(40)

and, as a consequence

$$\frac{d\mathbf{r}}{dt} = (x - x')\mathbf{i} \tag{41}$$

we get

$$\int_{C} \mathbf{F} \cdot \mathbf{dl} = \int_{0}^{1} F_{1}(x' + t(x - x'), y, z)(x - x')dt$$
(42)

and, let $\tilde{x} = x' + t(x - x')$ before renaming \tilde{x} to x.

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}.$$

Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that \mathbf{F} is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \ge \frac{1}{2}$. Is \mathbf{F} conservative in the domain defined by $x^2 + y^2 \ge \frac{1}{2}$, $y \ge 0$? If so obtain a scalar potential for \mathbf{F} .

Solution:

$$\nabla \times \mathbf{F} = \frac{1}{2} \mathbf{k} \left[\partial_x \left(\frac{-x}{x^2 + y^2} \right) + \partial_y \left(\frac{y}{x^2 + y^2} \right) \right] \\ = \frac{1}{2} \mathbf{k} \left[-\frac{1}{x^2 + y^2} + \frac{2x^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} \right] \\ 6$$

$$= 0.$$

To show that **F** is not conservative consider $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle. Using the obvious parametrization

(43)

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \int_0^{2\pi} du \left(-\sin^2 u - \cos^2 u \right)$$
$$= -2\pi \neq 0, \tag{44}$$

therefore \mathbf{F} is not conservative.

The domain $x^2 + y^2 \ge \frac{1}{2}$, $y \ge 0$ is simply connected and **F** is irrotational and smooth is the domain. Thus **F** is conservative.

Write $\mathbf{F} = \nabla \phi$. Seek a $\phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = \frac{y}{x^2 + y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{x}{x^2 + y^2}.$$
(45)

Integrate first equation by treating y as a constant

$$\phi(x,y) = y \int \frac{dx}{x^2 + y^2} = \tan^{-1}\frac{x}{y} + C(y).$$
(46)

Assume that x and y are non-negative, then

$$\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = \frac{\pi}{2},$$

so that $\phi(x, y) = -\tan^{-1} \frac{y}{x} + a$ possibly y-dependent constant. However it is easy to check that $\phi = -\tan^{-1} \frac{y}{x}$ satisfies $\frac{\partial \phi}{\partial y} = -\frac{x}{x^2+y^2}$. Clearly, $\tan^{-1} \frac{y}{x}$ is the usual polar angle θ , that is $\phi = -\theta$.

Can try to extend this back to the original domain $x^2 + y^2 \ge \frac{1}{2}$, but ϕ will suffer a branch cut discontinuity at, say $\theta = \frac{3}{2}\pi$.