231 Tutorial Sheet 18: soln to question 4[1,2]

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4. Use the recursion relation to show that the functions H_n defined through the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

satisfy Hermites equation

$$y'' - 2xy' + 2ny = 0.$$

Soln. So, the question gives the wrong hint, or even the wrong instruction, since it tells you to use the recursion relation. If it hadn't said that, an easier approach would be to do a direct substitution. Hence

$$\frac{\partial}{\partial x}\Phi(x,h) = 2he^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H'_n(x) \tag{1}$$

and

$$\frac{\partial^2}{\partial x^2}\Phi(x,h) = 4h^2e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H''_n(x) \tag{2}$$

So

$$\frac{\partial^2}{\partial x^2}\Phi(x,h) - 2x\frac{\partial^2}{\partial x^2}\Phi(x,h) = 4(h-x^2)he^{2xh-h^2} \tag{3}$$

The trick now is to spot[3] that the right hand side is

$$-4(h-x^2)he^{2xh-h^2} = 2h\frac{\partial}{\partial h}e^{2xh-h^2} = \sum_{n=0}^{\infty} n\frac{h^n}{n!} H_n(x) \tag{4}$$

Hence

$$\frac{\partial^2}{\partial x^2}\Phi - 2x\frac{\partial^2}{\partial x^2}\Phi + 2h\frac{\partial}{\partial h}\Phi = 0 \tag{5}$$

and writing this out in terms of the sums gives

$$\sum_{n=0}^{\infty} \frac{h^n}{n!}\left[H''_n(x) - 2xH'_n(x) + 2nH(x)\right] \tag{6}$$

and equating each coefficient of each power of h to zero gives the result.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Including material from Chris Ford, to whom many thanks.

[3]I have now noticed that when Chris set a question like this a few years ago he gave a hint like this

Doing the same question using the recursion relation is more complicated, you need to consider the expansion:

$$e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n(2x-h)^n}{n!} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) (2x)^{n-m}(-1)^m h^m \tag{7}$$

where the last expression comes from the binomial expansion. Now, we just need to do a change of index to get this into the form

$$e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x) \tag{8}$$

Lets start by setting $p = m + n$, the current index of h, the complication here is that n appears in the sum range of m, so the end point of the m sum is $m = p - m$; it is good to check by hand how the sums ranges change, but basically this means $m = p/2$ for p even and $m = (p-1)/2$ for m odd.

$$e^{2xh-h^2} = \sum_{p=0}^{\infty} \sum_{m=0}^{M(p)} \frac{1}{(p-m)!} \left(\begin{array}{c} p-m \\ m \end{array} \right) (2x)^{p-2m}(-1)^m h^p \tag{9}$$

where $M(p)$ denotes the correct end point for even and odd p. Now, we mess around a bit: lets concentrate on the even sum and let $p = 2q$ and $j = q - m$, so

$$e^{2xh-h^2} = \text{odd} + \sum_{q=0}^{\infty} \sum_{j=0}^{q} \frac{1}{(2j)!(q-j)!} 2^{2j} x^{2j} (-1)^j h^{2q} \tag{10}$$

and hence

$$H_{2q}(x) = \sum_{j=0}^{q} \frac{(2q)!}{(2j)!(q-j)!} 2^{2j} x^{2j} (-1)^j \tag{11}$$

giving

$$a_{2j} = \frac{(2q)!}{(2j)!(q-j)!} 2^{2j} (-1)^j \tag{12}$$

and

$$\begin{aligned} a_{2j+2} &= \frac{(2q)!}{(2j+2)!(q-j-1)!} 2^{2j+2} (-1)^{j+1} && (13) \\ &= -4 \frac{q-j}{(2j+2)(2j+1)} a_{2j} = 2 \frac{2j-2q}{(2j+2)(2j+1)} a_{2j} && (14) \end{aligned}$$

which, since $\alpha = 2q$ and n in the notes is our $2j$, after all these changes of index, is what we want!