231 Outline Solutions Tutorial Sheet 16, 17 and 18.[1][2]

13 May 2007

Problem Sheet 16

1. Obtain the general solutions of the ODEs

   (a) $y'' + 5y' + 6y = e^x$

   (b) $y'' + 5y' + 6y = e^{-2x}$

   (c) $y'' + 5y' + 6y = \sinh x$

   *Solution:*Auxiliary equation gives $\lambda^2 + 5\lambda + 6 = 0$ roots $\lambda = -2$, $\lambda = -3$ so we get

   $$y(x) = C_1 e^{-2x} + C_2 e^{-3x} \tag{1}$$

   solving the corresponding homogeneous equation

   (a) Now substitute $y = Ce^x$ to get

   $$C + 5C + 6C = 1 \tag{2}$$

   so $C = 1/12$ and

   $$y(x) = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{12} e^x \tag{3}$$

   (b) This time the right hand side matches one of the complementary solutions, so we substitute $y = Cx \exp(-2x)$, the terms with xs outside the exponential all cancel and we get

   $$-4C - 10C = 1 \tag{4}$$

   or $C = -1/14$ giving

   $$y(x) = C_1 e^{-2x} + C_2 e^{-3x} - \frac{1}{14} x e^{-2x} \tag{5}$$

   (c) PI: $y'' + 5y' + 6y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. PI for $y'' + 5y' + 6y = \frac{1}{2}e^x$, $y = Ce^x$ $C(1 + 5 + 6) = \frac{1}{2}$, i.e. $C = \frac{1}{24}$. PI for $y'' + 5y' + 6y = -\frac{1}{2}e^{-x}$, $y = Ce^{-x}$ so that $C(1 - 5 + 6) = -\frac{1}{2}$, i.e. $C = -\frac{1}{4}$. PI for full problem $y_p(x) = \frac{1}{24}e^x - \frac{1}{4}e^{-x}$.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Including material from Chris Ford, to whom many thanks.

2. Obtain the general solution of the ODE

$$y''(x) + 3y'(x) + 3y(x) = f(x)$$

where f is the periodic function defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

*Solution:*CF: auxiliary equation $\lambda^2 + 3\lambda + 3 = 0$ with roots $\lambda = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$ giving

$$y_c(x) = e^{-\frac{3}{2}x}\left(A\cos\frac{\sqrt{3}}{2} + B\sin\frac{\sqrt{3}}{2}x\right).$$

PI: use Fourier expansion of f

$$f(x) = -\frac{4}{\pi}\sum_{n \text{ odd } >0}\frac{\cos nx}{n^2} = -\frac{2}{\pi}\sum_{n \text{ odd } \in Z}\frac{e^{inx}}{n^2}.$$

PI for $y'' + 3y' + 3y = e^{inx}$; trying $y = Ce^{inx}$ gives $(-n^2 + 3in + 3)C = 1$. PI for full problem

$$y_p = -\frac{2}{\pi}\sum_{n \text{ odd } \in Z}\frac{e^{inx}}{n^2(-n^2 + 3in + 3)}.$$

3. Obtain the general solutions of the ODEs

   (a) $y'' + y' + 3y = 0$

   (b) $y'' + y = f(x)$, where f is the periodic square wave defined by

   $$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \quad \text{and } f(x + 2\pi) = f(x)$$

   (c) $y'' + y' + 3y = e^{-|x|}$.

   Solution:

   (a) $y'' + y' + 3y = 0$. Auxiliary equation: $\lambda^2 + \lambda + 3 = 0$ with roots $\lambda = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{11}$.

   $$y = e^{-\frac{1}{2}x}\left(A\cos(\frac{1}{2}\sqrt{11}x) + B\sin(\frac{1}{2}\sqrt{11}x)\right).$$

   (b) $y''(x) + y(x) = f(x)$. CF: $y_c = A\sin x + B\cos x$. To find the PI write $f(x)$ as a Fourier series (obtained in the lectures)

   $$f(x) = \frac{4}{\pi}\sum_{n>0,\ n \text{ odd}}\frac{1}{n}\sin nx = \frac{2}{\pi i}\sum_{n \text{ odd}}\frac{e^{inx}}{n}.$$

First find a PI for $y'' + y = e^{inx}$. Trying $y = Ce^{inx}$ gives $(-n^2+1)C = 1$ so that $C = 1/(1-n^2)$ unless $n = \pm 1$.
$n = 1$: Try $y = Cxe^{ix}$ so that $y'' = (-xe^{ix} + 2ie^{ix})C$. Therefore a particular solution to $y'' + y = e^{ix}$ is $y = -\frac{1}{2}ixe^{ix}$.
$n = -1$: Similarly particular solution to $y'' + y = e^{-ix}$ is $\frac{1}{2}ixe^{-ix}$
PI for full problem

$$\begin{aligned} y_p &= \frac{2}{\pi i} \sum_{n \text{ odd},\, n \neq \pm 1} \frac{e^{inx}}{n(1-n^2)} + \frac{2}{\pi i} \cdot -\frac{1}{2} ixe^{ix} - \frac{2}{\pi i} \cdot \frac{1}{2} ixe^{-ix} \\ &= \frac{4}{\pi} \sum_{n > 2 \text{ odd}} \frac{\sin nx}{n(1-n^2)} - \frac{2x}{\pi} \cos x. \end{aligned}$$

(c) CF: general solution of part (a), PI: write $f(x) = e^{-|x|}$ as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k) = \int_{-\infty}^{\infty} dk\ e^{ikx} \frac{1}{\pi} \frac{1}{1+k^2}.$$

PI for $y'' + y' + 3y = e^{ikx}$. Trying $y = Ce^{ikx}$ gives $C(-k^2 + ik + 3)e^{ikx} = e^{ikx}$ so that $C = 1/(-k^2 + ik + 3)$. Therefore a PI to the full problem is

$$y_p(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk\ e^{ikx} \frac{1}{1+k^2} \frac{1}{(-k^2 + ik + 3)}.$$

Problem Sheet 17

1. Solve $x^2y'' + 4xy' + y = 0$. *Solution:*$x^2y'' + 4xy' + y = 0$. The standard substitution $x = e^z$ gives
$$\frac{d^2y}{d^2z} + (4-1)\frac{dy}{dz} + y = 0. \tag{6}$$
Auxiliary equation $\lambda^2 + 3\lambda + 1 = 0$ with roots $\lambda = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$
$$y = C_1 e^{-(\frac{3}{2} - \frac{1}{2}\sqrt{5})z} + C_2 e^{-(\frac{3}{2} + \frac{1}{2}\sqrt{5})z} = C_1 x^{-(\frac{3}{2} + \frac{1}{2}\sqrt{5})} + C_2 x^{-(\frac{3}{2} - \frac{1}{2}\sqrt{5})}. \tag{7}$$

2. Solve $x^2y'' + 4xy' + y = x^5$. *Solution:*So the same substitution gives
$$\frac{d^2y}{d^2z} + 3\frac{dy}{dz} + y = e^{5z} \tag{8}$$
So, we already have the complementary function for this, we just need the particular integral, substitute $y = C\exp(5z)$ giving
$$25C + 15C + C = 1 \tag{9}$$

giving

$$y = C_1 x^{-(\frac{3}{2}+\frac{1}{2}\sqrt{5})} + C_2 x^{-(\frac{3}{2}-\frac{1}{2}\sqrt{5})} + \frac{1}{41}x^5. \tag{10}$$

This isn't such a good question, the one I meant to ask was something like

$$x^2y'' - 3xy' - 5y = x^5 \tag{11}$$

After substitution this gives

$$\frac{d^2y}{d^2z} - 4\frac{dy}{dz} - 5y = e^{5z} \tag{12}$$

So, the complementary equation is

$$\lambda^2 + 5\lambda + 5 = 0 \tag{13}$$

leading to $\lambda = 5$ or $\lambda = -1$. Now, to get the particular integral, we need to substitute $y = Cz\exp(5z)$. Hence

$$10C - 4C = 1 \tag{14}$$

so $C = 1/6$ and

$$y = C_1x^5 + \frac{C_2}{x} + \frac{1}{6}x^5 \log x \tag{15}$$

Another question along the same lines would be

$$x^2y'' + 3xy' + y = 0 \tag{16}$$

Here substituting $x = e^z$ yields

$$\frac{d^2y}{d^2z} + (3-1)\frac{dy}{dz} + y = 0.$$

Auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = -1$ so that $y(x) = C_1e^{-z} + C_2ze^{-z} = C_1x^{-1} + C_2x^{-1}\log x$.

Problem Sheet 18

1. Use the recursion relation

$$a_{n+2} = \frac{2(n-\alpha)a_n}{(n+1)(n+2)}$$

or the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!}H_n(x)$$

to obtain polynomial solutions of Hermite's equation $y'' - 2xy' + 2\alpha y = 0$ for $\alpha = 3$, 4 and 5.

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*Solution:*Ok lets use the generating function, so, we want everything up to h^5

$$\begin{aligned}\Phi(x,h) &= e^{2xh-h^2} \\ &= 1 + (2xh - h^2) + \frac{1}{2}(2xh-h^2)^2 + \frac{1}{6}(2xh-h^2)^3 + \frac{1}{24}(2xh-h^2)^4 + \frac{1}{120}2^5x^5h^5 + O(h\end{aligned}$$

and, continuing to drop high powers in h

$$\Phi(x,h = 1 + 2xh - h^2 + \frac{1}{2}(4x^2h^2 - 4xh^3 + h^4) + \frac{1}{6}(8x^3h^3 - 12x^2h^4 + 6xh^5) + \frac{1}{24}(16x^4h^4 - 32$$

Hence

$$\begin{aligned}P_3(x) &= 8x^3 - 12x \\ P_4(x) &= 16x^4 - 48x^2 + 12 \\ P_5(x) &= 32x^5 - 160x^3 + 120x \qquad (19)\end{aligned}$$

3. Legendre's equation can be written

$$(1-x^2)y'' - 2xy' + \alpha y = 0,$$

where α is a constant. Consider a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine a recursion relation for the a_n coefficients. For what values of α does Legendre's equation have polynomial solutions?

Solution:$y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ Therefore $xy'(x) = \sum_{n=0}^{\infty} n a_n x^n$ and $x^2 y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^n$.

Relabel $y''(x)$ $(n = m+2)$

$$y''(x) = \sum_{m=-2}^{\infty} a_{m+2}(m+2)(m+1)x^m = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m.$$

The last step used that the first two terms in the sum are zero. The ODE can be written

$$(1-x^2)y''(x)-2xy'(x)+\alpha y(x) = \sum_{m=0}^{\infty} x^m \left[a_{m+2}(m+2)(m+1) - a_m m(m-1) - 2ma_m + \alpha a_m\right] = 0.$$

giving the recursion relation

$$a_{m+2} = \frac{m(m+1)-\alpha}{(m+2)(m+1)} a_m.$$

If α is of the form $n(n+1)$ $(n = 0, 1, 2, ...)$ *one* of the solutions of the ODE will be a polynomial since the recursion relation will terminate.

4. (Frobenius training exercise) For each of the following equations obtain the indicial equation for a Frobenius series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

   (a) $y'' + y = 0.$

   (b) $x^2y'' + 3xy' + y = 0$

   (c) $4xy'' + 2y' + y = 0.$

In case a) use the method of Frobenius to obtain the general solution. In case b) use the method of Frobenius to find one solution (the method fails to give the other solution).

Solution:(a) Frobenius: $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ where s is to be determined. Differentiating twice gives $y''(x) = \sum(n+s)(n+s-1)x^{n+s-2}$ (most singular term in ODE). Relabel this as $y''(x) = a_0 s(s-1)x^{s-2} + a_1(s+1)sx^{s-1} + \sum_{m=0}^{\infty} a_{m+2}(m+s+2)(m+s+1)x^{m+s}$ and so

$$\begin{aligned} y''(x) + y(x) &= a_0 s(s-1)x^{s-2} + a_1(s+1)sx^{s-1} \\ &+ \sum_{m=0}^{\infty} x^{m+s}\left[a_{m+2}(m+s+2)(m+s+1) + a_m\right] = 0. \end{aligned}$$

Take $a_0 = 1$. The indicial equation is

$$s(s-1) = 0,$$

with roots $s = 0$ and $s = 1$.

<u>$s = 0$</u> For this s can take $a_1 \neq 0$ but for now set $a_1 = 0$.

Recursion relation: $a_{n+2}(n+2)(n+1)+a_n=0$ or

$$a_{n+2}=-\frac{a_n}{(n+2)(n+1)},$$

which is particularly easy to solve $a_0=1$, $a_2=-\frac{1}{1\cdot 2}$, $a_4=\frac{1}{1\cdot 2\cdot 3\cdot 4}$, ..., $a_{2p}=\frac{(-1)^p}{(2p)!}$. The solution is

$$y(x)=\sum_{p=0}^{\infty}\frac{(-1)^p x^{2p}}{(2p)!}=\cos x.$$

Including $a_1\neq 0$ gives $y(x)=\cos x+a_1\sin x$. The $s=1$ solution is also $y(x)=\sin x$. The general solution is

$$y(x)=A\cos x+B\sin x.$$

(b) $x^2y''+3xy'+y=0$. Frobenius $y=\sum_{n=0}a_nx^{n+s}$, $xy'=\sum_{n=0}a_n(n+s)x^{n+s}$, $x^2y''=\sum_{n=0}a_n(n+s)(n+s-1)x^{n+s}$. No 'most singular' term or terms! No recursion relation! Indicial equation? (need a_0 contributions) $x^2y''+3xy'+y=a_0x^s\left[s(s-1)+3s+1\right]+\text{higher powers}=0$ so that $s^2+2s+1=0$ with two equal roots $s=-1$. Since there is no recursion relation the a_n $(n>0)$ are all zero. $y(x)=x^{-1}$ is one solution (the other solution is not a Frobenius series).

(c) Write $y=\sum_{n=0}^{\infty}\ a_n\ x^{n+s}$
$y'=\sum_{n=0}^{\infty}\ a_n\ (n+s)\ x^{n+s}$, $\quad y''=\sum_{n=0}^{\infty}\ a_n\ (n+s)(n+s-1)\ x^{n+s}$. Now:
$y'(x)=a_0sx^{s-1}+\sum_{n=0}^{\infty}a_{n+1}(n+1+s)x^{n+s}$
$xy''(x)=a_0s(s-1)x^{s-1}+\sum_{m=0}^{\infty}a_{m+1}(m+1+s)(m+s)x^{m+s}$.

$$\begin{aligned}4xy''+2y'+y &= a_0\left[4s(s-1)+2s\right]\\ &\quad+\sum_{m=0}^{\infty}\left[4(m+1+s)(m+s)a_{m+1}+2(m+1+s)a_{m+1}+a_m\right]x^{m+s}\\ &\quad 4a_0s(s-\frac{1}{2})x^{s-1}+\sum_{m=0}^{\infty}\left[4(m+1+s)(m+s+\frac{1}{2})a_{m+1}+a_m\right].\end{aligned}$$

Set $a_0=1$ Indicial equation: $s(s-\frac{1}{2})=0$ with roots $s=0$ and $s=\frac{1}{2}$.

5. Use the recursion relation to show that the functions H_n defined through the generating function

$$\Phi(x,h)=e^{2xh-h^2}=\sum_{n=0}^{\infty}\frac{h^n}{n!}H_n(x)$$

satisfy Hermites equation

$$y''-2xy'-2ny=0.$$

*Solution:*Done separately.