231 Outline Solutions Tutorial Sheet 13, 14 and 15.[1][2]

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Problem Sheet 13

1. Express the following functions as Fourier integrals:

   (a)
$$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

   (b)
$$f(x) = \frac{\sin x}{x}$$

Solution:(a) Writing f as a Fourier integral $f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k)$. We require the Fourier transform:

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{-ikx}\ f(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk\ e^{-ikx}\ \frac{e^{ix}+e^{-ix}}{2} \\
&= \frac{1}{4\pi}\left(\frac{e^{i(1-k)x}}{i(1-k)} + \frac{e^{i(-1-k)x}}{i(-1-k)}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-ik\pi/2}+ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2}-ie^{ik\pi/2}}{i(-1-k)}\right] \\
&= \frac{1}{4\pi}\ 2\cos\left(\frac{k\pi}{2}\right)\ \left(\frac{1}{1-k}+\frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right)\ \frac{1}{1-k^2}.
\end{aligned}$$

Therefore
$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\ \cos\left(\frac{k\pi}{2}\right)\ \frac{e^{ikx}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$. (b)

$$\frac{\sin x}{x} = \frac{1}{2}\int_{-1}^{1} dk\ e^{ikx}.$$

Remark: In the lectures it was shown that the Fourier transform of a square pulse is proportional to $\sin k/k$ and so it follows that the Fourier transform of the $\sin x/x$ is proportional to the pulse and, for example, integrating quickly gives the constant of proportionality.

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[2]Including material from Chris Ford, to whom many thanks.

2. Prove the following properties of the Fourier transform

   (a) The Fourier transform of an even function is even.
   (b) The Fourier transform of a real odd function is purely
   (c) $\tilde{f}'(k) = ik\tilde{f}(k)$.
   (d) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

Solution:(a) Assume that f is even, i.e. $f(-x) = f(x)$, then

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikx} f(x).$$

make the change of variables $y = -x$:

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{-iky} f(-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{-iky} f(y) = \tilde{f}(k).$$

(b) Assume that f is real and odd, i.e. $f(-x) = -f(x)$ and $\bar{f}(x) = f(x)$

$$\overline{\tilde{f}(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikx}\ f(x).$$

Make the change of variables $y = -x$

$$\overline{\tilde{f}(k)} = \frac{1}{2\pi}\int_{\infty}^{-\infty} (-dy)\ e^{-iky} f(-y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} (dy)\ e^{-iky} f(y) = -\tilde{f}(k).$$

(c) here an integration by parts is required

$$\tilde{f}'(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-ikx} f'(x) = e^{-ikx} f(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx(-ik)e^{-ikx} f(x) = ik\tilde{f}(k),$$

assuming that the boundary terms vanish.
(d) The Fourier integral representation of a function f, i.e.

$$f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k),$$

can be written as

$$f(-x) = 2\pi\tilde{\tilde{f}}(x),$$

or $\tilde{\tilde{f}}(x) = f(-x)/(2\pi)$. Acting with the Fourier transform twice reproduces the original function up to the $x \to -x$ reflection multiplied by $1/(2\pi)$. Acting with the Fourier transform four times reproduces the original function multiplied by $(2\pi)^{-2}$.

3. Compute

(a)
$$\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3)$$

(b)
$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x)$$

(c)
$$\int_{0}^{2} dx\ e^x\ \delta'(x-1)$$

(d)
$$\int_{0}^{\infty} dx\ e^{-ax}\delta(\cos x)$$

(e)
$$\int_{0}^{\infty} dx\ \delta(e^{ax}\cos x).$$

(f)
$$\frac{d}{dx}e^{a\theta(x)}.$$

where a is a constant.

Solution:(a) $\int_{-\infty}^{\infty}\ dx\ x^2\delta(x-3) = 3^2 = 9.$
(b)Use
$$\delta\left(h(x)\right) = \sum_i \frac{\delta(x-x_i)}{|h'(x_i)|},$$
where the x_i are roots of h. Here $h(x) = x^2+x = x(x+1)$ with roots $x_1 = 0$ and $x_2 = -1$. $h'(x) = 2x+1$ and so $h'(0) = 1$, $h'(-1) = -1$. This gives $\delta(x^2+x) = \delta(x)+\delta(x+1)$
$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x) = 2.$$
(c) Integrate by parts:
$$\int_0^2 dx\ e^x\ \delta'(x-1) = e^x\delta(x-1)|_0^2 - \int_0^2 dx\ e^x\ \delta(x-1) = -e.$$
(d) $h(x) = \cos x$ has zeros at $x = \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi$ etc. and the derivative of $\cos x$ is equal to 1 or -1 at these points. Therefore
$$\int_0^{\infty} dx\ e^{-ax}\delta(\cos x) = \sum_{n=0}^{\infty} e^{-a(\frac{1}{2}\pi+n\pi)} = e^{-\frac{1}{2}a\pi}\sum_{n=0}^{\infty} e^{-an\pi} = \frac{e^{-\frac{1}{2}a\pi}}{(1-e^{-a\pi})},$$

the last step used the standard geometric series formula. The result may be rewritten in terms of the hyperbolic sine.

$$\int_0^\infty dx\ e^{-ax}\delta(\cos x) = \frac{1}{2\sinh\frac{1}{2}a\pi}.$$

(e) $h(x) = e^{ax}\cos x$, $h'(x) = ae^{ax}\cos x - e^{ax}\sin x$. The zeros of h are the same as in the previous problem. At a zero $|h'(x)| = e^{ax}$. This implies that the integral leads to the same geometric sum as in part (d). (f) First you need to reexpress everything so that it is linear in $\theta(x)$, we can't differenciate powers of $\theta(x)$. So

$$\begin{aligned}\exp a\theta &= \sum_{n=0}^{\infty}\theta(x)^n a^n n! \\ &= 1 + \sum_{n=1}^{\infty}\theta(x)^n a^n n!\end{aligned}$$

then, using $\theta^n = \theta$, easy to check from the definition of θ, we get

$$\begin{aligned}\exp a\theta &= 1 + \theta\sum_{n=1}^{\infty}\frac{a^n}{n!} \\ &= 1 + \theta\left(\sum_{n=0}^{\infty}\frac{\theta(x)^n a^n}{n!} - 1\right) \\ &= 1 + \theta\left(e^a - 1\right)\end{aligned}$$

and hence

$$\frac{d}{dx}e^{a\theta(x)} = \frac{d}{dx}\left[1 + \theta\left(e^a - 1\right)\right] = \delta(x)\left(e^a - 1\right)$$

Problem Sheet 14

1. In the lectures (quite a while ago) it was shown that the scalar field

$$\phi(\mathbf{r}) = \frac{1}{r},$$

where $r = \sqrt{x^2+y^2+z^2}$ is harmonic except at the origin. In fact it can be shown that

$$\nabla^2\phi(\mathbf{r}) = -4\pi\delta^3(\mathbf{r}). \qquad (A)$$

Formally apply Gauss' theorem to the vector field $\mathbf{F} = \nabla\phi$ to show that

$$\int_{r<a} dV\ \nabla^2\ \phi = -4\pi.$$

This is clearly consistent with (A). Another treatment would replace the singular scalar field ϕ with a sequence of smooth scalar fields, e.g.

$$\phi_n(\mathbf{r}) = \frac{n}{\sqrt{n^2r^2+1}}.$$

Prove that

$$\int_{R^3} dV\ \nabla^2\phi_n(\mathbf{r}) = -4\pi.$$

Solution: $\mathbf{F} = \nabla\phi$ so that $\nabla^2\phi =$ div $\mathbf{F}$. Applying Gauss' theorem

$$\int_{r<a} dV\ \nabla^2\phi = \int_{r<a} dV\ \text{div}\ \mathbf{F} = \int_{r=a} \mathbf{F}\cdot\mathbf{dA}.$$

$\mathbf{F} = -\mathbf{r}/r^3$ and $\mathbf{F}\cdot\mathbf{n} = -1/a^2$ and the surface are is $4\pi a^2$ giving

$$\int_{r<a} dV\ \nabla^2\phi = -4\pi.$$

$\partial_x\phi_n = -\frac{1}{2}n(n^2r^2+1)^{-3/2}2xn^2$, and similarly for $\partial_y\phi_n$ and $\partial_z\phi_n$. Therefore

$$\nabla\phi_n = -\frac{n^3\mathbf{r}}{(n^2r^2+1)^{3/2}}.$$

$$\int_{r<a} dV\ \nabla^2\phi_n = \int_{r=a} \nabla\phi_n\cdot\mathbf{dA} = -\frac{n^3\ 4\pi a^3}{(n^2a^2+1)^{3/2}} \to -4\pi$$

as $a\to\infty$.

Problem Sheet 15

1. Inside an integral, what is
$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} \tag{1}$$
for $\theta(x)$ the usual Heaviside function and $|\epsilon|<1$.

*Solution:*There are two ways to do this, we can either expand the fraction as a power series or we can try and evaluate it inside an integral. First the first way, using the usual expansion of $1/(1+x)$ for $x<1$
$$\frac{1}{1+\epsilon\theta(x)}=\sum_{n=0}^{\infty}[-\epsilon\theta(x)]^n \tag{2}$$
Now, we just use the fact that $\theta(x)^n=\theta(x)$ for n a positive integer; note that we have to be careful with the first term in the series which doesn't contain a $\theta(x)$ factor since n is zero. Hence
$$\frac{1}{1+\epsilon\theta(x)}=1+\theta(x)\sum_{n=1}^{\infty}(-\epsilon)^n=1+\theta(x)\sum_{n=0}^{\infty}(-\epsilon)^n-\theta(x)=1-\theta(x)+\frac{1}{1+\epsilon}\theta(x) \tag{3}$$
where we have added and taken away the missing term in the sum. Hence,
$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)}=\frac{d}{dx}[1-\theta(x)+\frac{1}{1+\epsilon}\theta(x)]=\frac{1}{1+\epsilon}\delta(x)-\delta(x) \tag{4}$$
The other way is to do the calculation inside an integral: with $a<0$ and $b>0$
$$I=\int_a^b dxf(x)\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} \tag{5}$$
Now, integrating by parts
$$I=f'(x)\frac{1}{1+\epsilon\theta(x)}\Big]_a^b-\int_a^b dxf'(x)\frac{1}{1+\epsilon\theta(x)} \tag{6}$$
Next, we split the integral into two and use the fact $\theta(x)$ is zero for negative x and one for positive x. So,
$$I=\frac{1}{1+\epsilon}f(b)-f(a)-\int_a^0 dxf'(x)-\frac{1}{1+\epsilon}\int_0^b dxf'(x) \tag{7}$$
and, using the Fundamental Theorem of Calculus
$$I=\frac{1}{1+\epsilon}f(b)-f(a)-f(0)+f(a)-\frac{1}{1+\epsilon}f(b)+\frac{1}{1+\epsilon}f(0)=\left(\frac{1}{1+\epsilon}-1\right)f(0) \tag{8}$$
which implies
$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)}=\frac{1}{1+\epsilon}\delta(x)-\delta(x) \tag{9}$$
as before.

2. Compute

   (a) $\int_{-\infty}^{\infty} dx\ e^x\ \delta(x+1)$
   (b) $\int_{-3}^{1} dx\ \delta(x^2-3x+2)$
   (c) $\int_{-\infty}^{\infty} dx\ \cos x\ \delta'(x)$
   (d) $\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right)$.

   Solution:

   (a) $\int_{-\infty}^{\infty} dx\ e^x\ \delta(x+1) = e^{-1}$.
   (b) Use
   $$\delta\left(h(x)\right) = \sum_i \frac{\delta(x-x_i)}{|h'(x_i)|},$$
   where the x_is are roots of h. In this case $h(x) = x^2-3x+2 = (x-2)(x-1)$ with roots $x_1 = 2$ and $x_2 = 1$. This is a problem since $x = 1$ is one of the limits of integration, in fact
   $$\int_{-\infty}^{0} dx\delta(x) \tag{10}$$
   isn't defined, and so the answer here is that the integral isn't defined. Say instead we had been asked
   $$\int_{-3}^{3} dx\ \delta(x^2-3x+2) \tag{11}$$
   then both roots are in the integral and we would use $h'(x) = 2x-3$ so that $h'(1) = -1$, giving $|h(1)| = 1$ and $h'(2) = 1$ which gives
   $$\delta(x^2-3x+2) = \delta(x+1) + \delta(x_2)$$
   and
   $$\int_{-3}^{3} dx\ \delta(x^2-3x+2) = 2.$$
   (c)
   $$\int_{-\infty}^{\infty} dx\ \cos x\delta'(x) = -\int_{-\infty}^{\infty} dx\ (-\sin x)\ \delta(x) = 0$$
   Integrating by parts and using $\sin 0 = 0$.
   (d) Use formula for $\delta(h(x))$, here $h(x) = \sin(1/x)$ which is zero for $1/x = n\pi$ ($n \in Z$). $h'(x) = -x^{-2}\cos(1/x)$ and since $|\cos n\pi| = 1$
   $$\delta\left(h(x)\right) = \sum_{n\neq 0} \frac{\delta\left(x-\frac{1}{n\pi}\right)}{\pi^2 n^2}.$$

Now $1/(n\pi) \in (0,1)$ for all positive n which gives

$$\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right) = \frac{1}{\pi^2}\sum_{n>0}\frac{1}{n^2}.$$

The sum on the RHS is $\zeta(2) = \pi^2/6$ (see Q3 Sheet 11) and so

$$\int_0^1 dx\ \delta\left(\sin\frac{1}{x}\right) = \frac{1}{6}.$$

3. Obtain a general solution to

   (a) $y' - 3y = e^{-x}$

   (b) $y' + y\cot x = \cos x$

   (c) $(x+1)y' + y = (x+1)^2$

   Solution:

   (a) Rewrite as

   $$e^{-3x}y' - 3ye^{-3x} = e^{-4x}$$

   or

   $$\left(e^{-3x}y\right)' = e^{-4x}$$

   and then integrate.

   (b) the quuickest thing to do is multiply across by the sine

   $$\sin xy' + cosxy = \sin x\cos x \tag{12}$$

   and rewritting

   $$(\sin xy)' = (\sin^2 x)' \tag{13}$$

   hence

   $$\sin xy = \sin^2 x + C \tag{14}$$

   or

   $$y = \sin x + C\operatorname{cosec} x \tag{15}$$

   (c) $(x+1)y' + y = (x+1)^2$ can again be rewritten

   $$[(x+1)y]' = x^2 + 2x + 1 \tag{16}$$

   so

   $$(x+1)y = \frac{1}{3}x^3 + x^2 + x + C \tag{17}$$

   or

   $$3y = \frac{x^3 + 3x^2 + 3x + 1}{x+1} + \frac{C}{x+1} = (x+1)^2 + \frac{C}{x+1} \tag{18}$$

   with a redefinition of C to get the nice devision at the end, another way to do this would have been to change variables to $z = x + 1$ at the start.

4. Obtain the general solutions of the following ODEs:

   (a) $y'' + 5y' + 6y = 0$
   (b) $y'' - 2y' + y = 0$

   Solution:

   (a) $y'' + 5y' + 6y = 0$ so substitute $e^{\lambda x}$ to get the auxiliary equation

   $$\lambda^2 + 5\lambda + 6 = 0 \tag{19}$$

   so $\lambda = -2$ and $\lambda = -3$ giving solution

   $$y = C_1 e^{-2x} + C_2 e^{-3x} \tag{20}$$

   (b) $y'' - 2y' + y = 0$ gives auxiliary equation

   $$\lambda^2 - 2\lambda + 1 = 0 \tag{21}$$

   which has $\lambda = 1$ as a repeated root, so

   $$y = C_1 e^{x} + C_2 x e^{x} \tag{22}$$